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Modeling and Measurements of CMUTs with Square Anisotropic Plates

Mette Funding la Cour, Thomas Lehrmann Christiansen, Christian Dahl-Petersen, Kasper Reck, Ole Hansen, Jørgen Arendt Jensen and Erik Vilain Thomsen

Center for Fast Ultrasound Imaging,
Biomedical Engineering Group,
Department of Electrical Engineering
Ørsteds Plads Building 349,
Technical University of Denmark,
2800 Kgs. Lyngby, Denmark.

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Modeling and Measurements of CMUTs with Square Anisotropic Plates

Mette Funding la Cour*†, Thomas Lehrmann Christiansen†, Christian Dahl-Petersen†, Kasper Reck†, Ole Hansen†, Jørgen Arendt Jensen* and Erik Vilain Thomsen†

*Center for Fast Ultrasound Imaging, Department of Electrical Engineering, Technical University of Denmark, DK-2800 Kgs. Lyngby, Denmark

†Department of Micro and Nanotechnology, Technical University of Denmark, DK-2800 Kgs. Lyngby, Denmark

***Abstract*—The conventional method of modeling CMUTs use the isotropic plate equation to calculate the deflection, leading to deviations from FEM simulations including anisotropic effects of around 10% in center deflection. In this paper, the deflection is found for square plates using the full anisotropic plate equation by use of the Galerkin method. Utilizing the symmetry of the silicon crystal, a compact and accurate expression for the deflection can be obtained. The deviation from FEM in center deflection is <0.1%. The deflection was measured on fabricated CMUTs using a white light interferometer. Fitting the anisotropic calculated deflection to the measurement a deviation of 0.5-1.5% is seen for the fitted values. Finally it was also measured how the device behaved under increasing bias voltage and it is observed that the model including anisotropic effects is within the uncertainty interval of the measurements.**

I. Introduction

Precise modeling of capacitive micromachined ultrasonic transducers (CMUT) is important for an efficient design process. The deflection $w(x,y)$ is an important parameter that influences several basic CMUT parameters such as pull in voltage and capacitance. Most existing analytical approaches use the isotropic plate equation to calculate the deflection [1], [2]. However, when using fusion bonding fabrication technology the plate usually consists of crystalline silicon, which is an anisotropic material. The isotropic approach is then invalidated and this results in deviations in the deflection compared to finite element modeling (FEM) and measurements. Therefore, to get precise modeling of these CMUTs the anisotropy of silicon needs to be taken into account.

For circular plates a simple and exact solution for the deflection exists, but this is not the case for square plates. Existing solutions to the deflection of square plates is based on series expansions with either trigonometric [3] or polynomial basis functions [4]. None of these, however, take the anisotropy of the plate into account.

Previously a model was made for calculating the deflection for an anisotropic plate with circular geometry [5], and in this paper the model is expanded to include square plates as well. The approach used to solve the full anisotropic plate equation is the Galerkin method [6]. Utilizing the symmetry of the silicon crystal, a compact and accurate approximation of the deflection can be obtained. The calculated deflection is compared to the solution for corresponding isotropic cases, a finite element model (FEM) and measurements performed on fabricated devices. Furthermore, the calculated deflection is used to find the stable position of the CMUT plate for a given bias voltage. Equivalent measurements are performed as well and the theory is compared to these.

II. The isotropic plate equation

Conventionally the deflection $w(x,y)$ of a CMUT with a thin plate is modeled using the isotropic plate equation [3]

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D_\mathrm{i}}, \tag{1}$$

where p is the applied pressure difference across the plate. The flexural rigidity is given by

$$D_\mathrm{i} = \frac{E}{12\,(1-\nu^2)} h^3 \tag{2}$$

with E being Young's modulus, ν being Poisson's ratio, and h being the thickness of the plate. For rectangular and square plates no simple exact solution exists to this equation and approximate methods have to be used. The traditional isotropic approach is based on a series expansion of the deflection and the center deflection for a thin clamped square plate having side length $2L$ is [3]

$$w_{0,\mathrm{isotropic}} = 0.0202448 \frac{L^4 p}{D_\mathrm{i}} \tag{3}$$

However, the plate material is often not isotropic and (1) and (2) are therefore no longer valid. Using the fusion bonding fabrication technique the plate usually consist of silicon which is an anisotropic material with a diamond cubic crystal structure. For plates made on silicon (111) substrates, Young's modulus and Poisson's ratio are constant and the isotropic plate equation can be used. However, for other silicon substrates, such as silicon (011) and silicon (001) which are most often used, Young's modulus and Poisson's ratio are strongly anisotropic and this leads to inaccurate deflection expressions.

III. Anisotropic plate equation

The solution is to use the generalized plate equation. This is a differential equation for the deflection, $w(x,y)$, of a thin anisotropic plate exposed to a uniform load p given by [7], [8]

$$\frac{\partial^4 w}{\partial x^4} + k_1 \frac{\partial^4 w}{\partial x^3 \partial y} + k_2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + k_3 \frac{\partial^4 w}{\partial x \partial y^3} + k_4 \frac{\partial^4 w}{\partial y^4} = \frac{p}{D_\mathrm{a}}. \tag{4}$$

TABLE I. ROOM TEMPERATURE (300K) STIFFNESS COEFFICIENTS FOR LOW DOPED N-TYPE CRYSTALLINE SILICON [10].

c^c_{11}	c^c_{12}	c^c_{44}
165.6 GPa	63.9 GPa	79.5 GPa

The plate coefficients k_1 to k_4 and the anisotropic flexural rigidity depend on the elastic constants of the plate material

$$k_1 = \frac{4C^{\text{eff}}_{16}}{C^{\text{eff}}_{11}} \quad k_2 = \frac{2\left(C^{\text{eff}}_{12}+2C^{\text{eff}}_{66}\right)}{C^{\text{eff}}_{11}} \quad k_3 = \frac{4C^{\text{eff}}_{26}}{C^{\text{eff}}_{11}}$$
$$k_4 = \frac{C^{\text{eff}}_{22}}{C^{\text{eff}}_{11}} \quad D_{\text{a}} = \frac{1}{12}h^3C^{\text{eff}}_{11} \tag{5}$$

where C^{eff}_{pq} are elements in the effective stiffness matrix. Notice that the stiffness of the plate is no longer expressed through Young's modulus and Poisson's ratio but directly through the stiffness values.

The stiffness matrix is the relation between stress and strain [9]

$$\sigma^{\text{c}} = \mathbf{c}^{\text{c}}\varepsilon^{\text{c}}, \text{ or } \varepsilon^{\text{c}} = \mathbf{s}^{\text{c}}\sigma^{\text{c}}. \tag{6}$$

Here superscript c denotes the crystallographic coordinate system, so $\mathbf{c}^{\text{c}}$ is the stiffness matrix and $\mathbf{s}^{\text{c}} = (\mathbf{c}^{\text{c}})^{-1}$ the compliance matrix in this coordinate system. Having a thin plate the stresses in the z direction can be ignored and plane stress assumed. Using the six vector notation, the relation between stress and strain becomes

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} C^{\text{eff}}_{11} & C^{\text{eff}}_{12} & C^{\text{eff}}_{16} \\ C^{\text{eff}}_{12} & C^{\text{eff}}_{22} & C^{\text{eff}}_{26} \\ C^{\text{eff}}_{16} & C^{\text{eff}}_{26} & C^{\text{eff}}_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{pmatrix}. \tag{7}$$

Depending on the symmetry of the crystal (7) can be further reduced. For silicon with a cubic crystal structure it becomes

$$\mathbf{C}^{\text{c}}_{\text{eff}} = \begin{pmatrix} c^{\text{c}}_{11} & c^{\text{c}}_{12} & 0 \\ c^{\text{c}}_{12} & c^{\text{c}}_{11} & 0 \\ 0 & 0 & c^{\text{c}}_{44} \end{pmatrix}. \tag{8}$$

The stiffness elements in this matrix are known from measurements and shown in Table I [10]. It is noted that all elements in (8) have the superscript c which means that they are known in the crystallographic coordinate system. However, the plate equation is valid in the plate coordinate system which is not necessarily the same. To illustrate this further the crystallographic and the plate coordinate systems can be seen in Fig. 1. The solid coordinate system along the [100] direction is where the stiffness values for silicon is known and the dashed system shows the rotated coordinate system for the plate where the stiffness values needs to be calculated. Having silicon as plate material and performing standard cleanroom fabrication, the plate will usually be on a (001) substrate and aligned to the wafer flat. Flat aligning is to the $\langle 110 \rangle$ direction and the plate coordinate system will be rotated $\psi = 45°$. A transformation of the stiffness matrix between the two coordinate systems is needed and the resulting effective stiffness matrix for the present case becomes [8]

$$C^{\text{eff}}_{[110]} = \begin{pmatrix} \frac{1}{2}(c^{\text{c}}_{11}+c^{\text{c}}_{12}+2c^{\text{c}}_{44}) & \frac{1}{2}(c^{\text{c}}_{11}+c^{\text{c}}_{12}-2c^{\text{c}}_{44}) & 0 \\ \frac{1}{2}(c^{\text{c}}_{11}+c^{\text{c}}_{12}-2c^{\text{c}}_{44}) & \frac{1}{2}(c^{\text{c}}_{11}+c^{\text{c}}_{12}+2c^{\text{c}}_{44}) & 0 \\ 0 & 0 & \frac{1}{2}(c^{\text{c}}_{11}-c^{\text{c}}_{12}) \end{pmatrix} \tag{9}$$

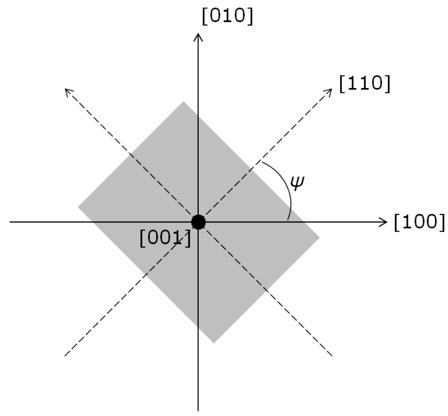

Fig. 1. The two coordinate systems, solid lines are the crystallographic system aligned to $\langle 100 \rangle$ direction and the dashed lines the plate system aligned to $\langle 110 \rangle$ direction.

TABLE II. SELECTED VALUES FOR THE PLATE COEFFICIENTS AND ANISOTROPIC FLEXURAL RIGIDITY FOR PLATES ON A SILICON (001) SUBSTRATE [8].

Orientation	ψ	k_1	k_2	k_3	k_4	$12D_a/h^3$[GPa]
[100]	0	0	2.81329	0	1	140.958
[110]	$\pi/4$	0	1.32413	0	1	169.618

It is seen that the plate now have an orthotropic structure. Using the stiffness elements from (9) in (5) it follows that $k_1 = k_3 = 0$ and $k_4 = 1$, so aligning the plate to the flat simplifies the plate equation (4) to

$$\frac{\partial^4 w}{\partial x^4} + k_2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D_{\text{a}}} \tag{10}$$

The same is the case for aligning the plate along the [100] direction where (8) is used giving the same values for k_1, k_3 and k_4. For these two special cases the coefficients in the plate equation is summerized in Tabel II.

IV. SOLVING THE PLATE EQUATION

Having a rectangular or square plate makes analytical deflection calculations complicated and approximate methods must be used to solve the generalized plate equation. With the anisotropic approach the Galerkin method [6] can be used to find approximate expressions for the deflection of a thin anisotropic square plate. In the most common case for CMUTs the plate is fabricated on a silicon (001) substrate and aligned to the [110] direction. For this orthotropic square plate with sidelengths $2L$ the relative deflection is found to [8], [11]

$$\frac{w}{w_0} = \left[1-\left(\frac{x}{L}\right)^2\right]^2 \left[1-\left(\frac{y}{L}\right)^2\right]^2 \times \left[1+\beta\left(\frac{x}{L}\right)^2+\beta\left(\frac{y}{L}\right)^2\right], \tag{11}$$

where the plate parameter is defined as

$$\beta = \frac{182+143k_2}{1432+91k_2} \tag{12}$$

Eqn. (11) and (12) is also valid for the plate aligned to the [100] direction on a silicon (001) substrate.

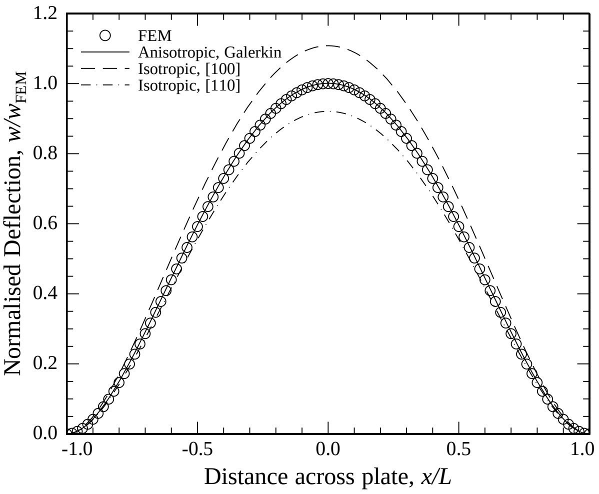

Fig. 2. Normalized deflection of a square plate of silicon (001) calculated using both the isotropic approach for [100] and [110] directions, using (15)-(13) and $k_2 = 2$ and the anisotropic approach using $k_2 = 1.32413$. The circles represent the deflection calculated by FEM.

The center deflection can be written

$$w_{0,\mathrm{Si}(001)} = \frac{77(1432+91k_2)}{256(16220+11k_2(329+13k_2))}\frac{L^4 p}{D_\mathrm{a}} \quad (13)$$

Note that the enter deflection depends only on the k_2 coefficient. For flat alignment it is found by inserting k_2 into (12) that $\beta = 0.23920$. This results in a normalized deflection surface for the plate aligned to the $\langle 110 \rangle$ direction given by

$$\left.\frac{w}{w_0}\right|_{\mathrm{Si}(001),\langle 110\rangle} = \left[1-(x/L)^2\right]^2\left[1-(y/L)^2\right]^2 \times \left[1+0.239207\left[(x/L)^2+(y/L)^2\right]\right] \quad (14)$$

and the center deflection becomes

$$w_0|_{\mathrm{Si}(001),\langle 110\rangle} = 0.0219611\frac{L^4 p}{D_a} \quad (15)$$

Comparing (3) and (15) it is seen that they are very similar containing the same parameters but different coefficients and the anisotropic instead of the isotropic flexural rigidity.

Fig. 2 shows a comparison between the analytical solution using the Galerkin method for a square plate of silicon (001), the isotropic solutions corresponding to [100] or [110] directions and finite element (FEM) simulations made using the full anisotropic stiffness matrix (using the stiffness coefficient in Table I) in COMSOL. The calculated deflections are normalized to the FEM center deflection. Excellent agreement is shown between the anisotropic case and FEM with a deviation of less than 0.1 % whereas the isotropic approach leads to deviations in the center deflection of around 10 % for both [100] and [110] directions.

V. CMUT APPLICATION

Many important design parameters for CMUTs depend on the deflection of the plate. By using static analysis it is possible to find the stable position of the plate when applying a certain bias voltage. The stable position is easiest expressed through the center deflection and is the position where the strain force balance the electrostatic and pressure forces. The center deflection is found from energy considerations. The total energy of the system consists of three terms:
1) Strain energy. Found by integrating the strain energy density

$$U_s = \frac{1}{2}\int_{-L}^{L}\int_{-L}^{L}\int_{-h/2}^{h/2}\varepsilon\cdot\sigma\ dxdydz. \quad (16)$$

The strain is calculated from (11) and the stress is found from (7) using the anisotropic effective stiffness matrix (9). The resulting strain energy becomes

$$\begin{aligned} U_{s,Si(001),[110]} &= \frac{4096h^3 w_0^2\left(632\beta^2+468\beta+1859\right)c_{11}}{4729725L^2} \\ &+ \frac{4096h^3 w_0^2\left(424\beta^2+468\beta+715\right)\left(c_{12}+2c_{44}\right)}{4729725L^2} \\ &= 4.520\times 10^{11}\frac{h^3 w_0^2}{L^2} \end{aligned} \quad (17)$$

2) Energy due to applied pressure. This is calculated from the pressure load on the plate

$$U_p = -\int_{-L}^{L}\int_{-L}^{L} pw\ dxdy \quad (18)$$

$$\begin{aligned} U_{p,Si(001),[110]} &= \frac{256L^2 p w_0(7+2\beta)}{1575} \\ &= -1.216 p w_0 L^2 \end{aligned} \quad (19)$$

3) Electrostatic energy. Found from the capacitance of the device

$$U_e = \frac{Q^2}{2C_t} \quad (20)$$

where the total capacitance C_t of the square plate is found using a Taylor expansion.
The total force on the system is then found by differentiating the total potential energy with respect to the center deflection. From this the stable center position of the plate can be found for a given applied voltage as the point where the total force is zero.

VI. COMPARISON TO MEASUREMENTS

CMUTs with square plates have been fabricated using fusion bonding. The fabricated devices have a 65x65 μm wide and 2.37 μm thick silicon plate with a gap height of 405 nm and a 198 nm thick insulating oxide at the bottom of the cavity. The deflection was measured with a Sensofar PLu Neox 3D Optical Profiler using white light interferometry.

Fig. 3 shows a measured cross section of the normalized deflection for the fabricated device. It is normalized in both center deflection and distance across the plate to compare the shape of the measured deflection with the calculated deflection. The red curve is a fit made to the measurements using the anisotropic model. As the cross section is taken through $y = 0$ the equation used for fitting is a reduced version of (11)

$$w_{fit} = w_0\left[1-(x/L)^2\right]^2\left[1+\beta(x/L)^2\right] \quad (21)$$

Both the center deflection and the plate parameter β is fitted. As it is seen in the figure the fitted value for β is 0.243 which

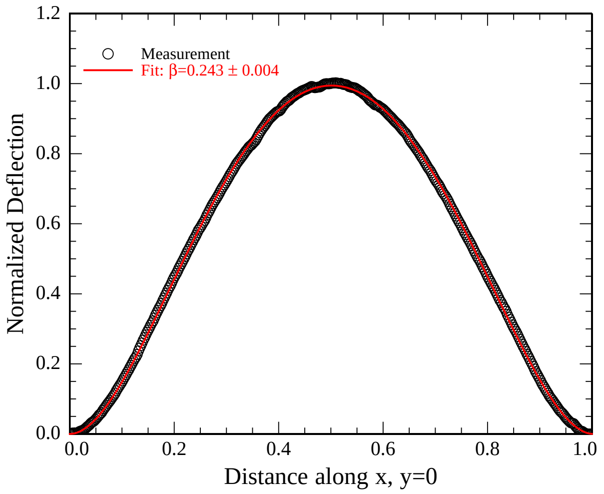

Fig. 3. Normalized deflection cross section from measurement on a fabricated CMUT. The red curve is a fit made with (21).

matches very well with a deviation of 1.5% from the calculated value of 0.23920 for this type of plate on silicon (001) substrate aligned to the [110] direction. The center deflection found from the fit has a deviation of 0.5% compared to the measurement.

Measurements with a DC voltage applied was also performed and the results is shown in Fig. 4. Here it is seen how the center deflection varies with the applied voltage and how it deflects more when approaching the pull-in voltage as expected. The center deflection for the measurements is found as the average of 10 cells. The errorbars corresponds to plus/minus two standard deviations. A theoretical curve made from the stable position analysis in section V is plotted as well. It is seen that the anisotropic theory matches well with the measurement as it is within the error margin. Also the pull-in voltage is in good agreement as it was measured to be 206 V for this design, compared to an expected value of 203 V from the anisotropic model.

VII. Conclusion

Due to the anisotropy of Young's modulus and Poisson's ratio for crystalline silicon, deviations between analytical models and FEM or measurements of up to 10 % is observed. It has been shown how to simplify the full anisotropic plate equation by utilizing the symmetry of the silicon crystal and how to solve the equation for a square plate using the Galerkin method. This results in a compact solution for thin square CMUT plates on a (001) silicon substrate aligned to the [110] direction (flat aligning), which predicts the deflection with an accuracy of less than 0.1% compared to FEM. Furthermore, the deflection was measured on fabricated devices and fitting the anisotropic calculated deflection to the measurement a deviation of 0.5-1.5% is observed in the fitted parameters. The stable position for varying bias voltage was also found using the anisotropic theory and comparing this to measurements it is seen that the theory is within the uncertainty interval of the measurements.

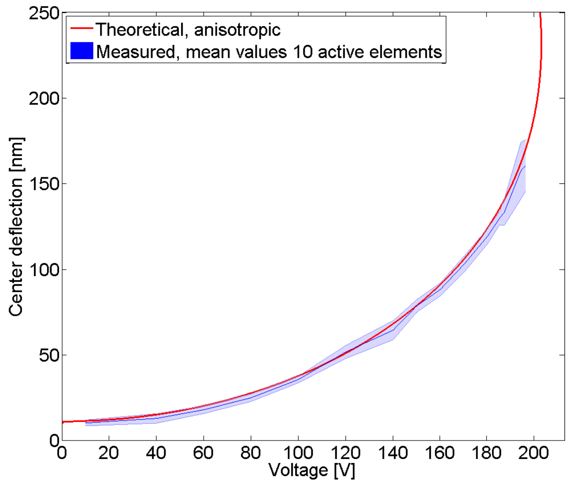

Fig. 4. Measured center deflection for increasing bias voltage. The solid red curve represents the anisotropic calculation of the stable position and the blue is the measurements.

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