Chapter 8

RANDOM SIGNALS

Signals can be divided into two main categories - deterministic and random. The term *random signal* is used primarily to denote signals, which have a random in its nature source. As an example we can mention the thermal noise, which is created by the random movement of electrons in an electric conductor. Apart from this, the term random signal is used also for signals falling into other categories, such as periodic signals, which have one or several parameters that have appropriate random behavior. An example is a periodic sinusoidal signal with a random phase or amplitude. Signals can be treated either as deterministic or random, depending on the application. Speech, for example, can be considered as a deterministic signal, if one specific speech waveform is considered. It can also be viewed as a random process if one considers the ensemble of all possible speech waveforms in order to design a system that will optimally process speech signals, in general.

The behavior of stochastic signals can be described only in the mean. The description of such signals is as a rule based on terms and concepts borrowed from probability theory. Signals are, however, a function of time and such description becomes quickly difficult to manage and impractical. Only a fraction of the signals, known as *ergodic*, can be handled in a relatively simple way. Among those signals that are excluded are the class of the non-stationary signals, which otherwise play an essential part in practice.

Working in frequency domain is a powerful technique in signal processing. While the spectrum is directly related to the deterministic signals, the spectrum of a random signal is defined through its correlation function. This is a "natural" consequence of the "uncertainty", which is characteristic to random signals.

Random signals can be both analog and digital. The values of digital signals are represented with a finite number of digits. This implies that the stochastic terms used are different for the two signal categories. In the following, however, we will assume that the values of digital signals are represented with infinite precision, and that we can describe the two types of signals in similar way. Often such signals are called "discrete time signals" rather than digital signals to emphasize the fact that the signal values are represented with infinite precision.

The main part of the signals that will be processed will be real. This does not mean that stochastic signals cannot be complex. Complex random signals can be analyzed the same way as real random signals with very few changes.

8.1 Random variables

In this section we set the framework for the description of the random processes and the subsequent signal processing. Regarding further details and proofs, the inquisitive reader is encouraged to read some of the standard texts on probability and statistics. We assume that the fundamental concepts of the probability theory, such as "experiment", "event", "outcome", etc. are familiar to the reader.

8.1.1 Definitions

In this section we will briefly review some of the key concepts and terms necessary to build the theory around stochastic signal processing.

In the study of probability, any process of observation is referred to as an experiment. The results of an observation are called *outcomes* of the experiment. If the outcomes of an experiment cannot be predicted, then it is called *random experiment*.

The set of possible outcomes of a random experiment is called the *sample space*. An element in the sample space is called a *sample point*. Each outcome of a random experiment corresponds to a sample point.

Subsets of the sample space are called *events*, and events consisting of a single element (sample point) are called *elementary events*.

Probability is a number, associated with events according to some appropriate *probability law*. The probability assigned to the event A from the sample space $\mathbb{S}$ $A \in \mathbb{S}$ is denoted as $P(A)$ and has a value between 0 and 1:

$$P(A), \qquad 0 \leqslant P(A) \leqslant 1$$

In order to be a valid probability assignment, the following three axioms must be satisfied:

1. $P(A) \geqslant 0$ for every event $A \in \mathbb{S}$.
2. $P(S) = 1$ for the certain event $\mathbb{S}$.
3. For any two mutually exclusive events A_1 and A_2,

$$P(A_1 \cup A_2) = P(A_1) + P(A_2).$$

By associating a number to the possible outcomes of an experiment, we define a *random variable*. A random variable can also be considered as a mathematical function that maps the outcomes of random experiment to numbers. For example, a random variable can be used to describe the process of rolling a fair die and the possible outcomes { 1, 2, 3, 4, 5, 6 }. Another random variable might describe the possible outcomes of picking a random person and measuring his or her height. Such random variables will be denoted with capital letters such as X, Y and Z. The set of values that are encountered in signal processing is either the set of real numbers $\mathbb{R}$ (analog signals), or the finite subset of rational numbers (digital signals).

In signal processing applications, it is the probabilistic description of the random variable, rather than the statistical characterization of events in the sample space, that is generally of interest. It is therefore more convenient to have a probability law assigned to the random variable itself. For a real-valued random variable X, one such statistical characterization is the *probability distribution function*, given by:

$$\boxed{W_X(\xi_1) = P\{X \leqslant \xi_1\}} \tag{8.1.1}$$

It is a function of the parameter ξ, and ξ_1 is a fixed value of the parameter. It follows directly from the definition that

$$0 \leqslant W_X(\xi) \leqslant 1, \tag{8.1.2}$$

and that $W_X(\xi)$ never decreases. We have also that:

$$P\{\xi_1 < X \leqslant \xi_2\} = W_X(\xi_2) - W_X(\xi_1), \tag{8.1.3}$$

as shown in the example in Fig. 8.1.

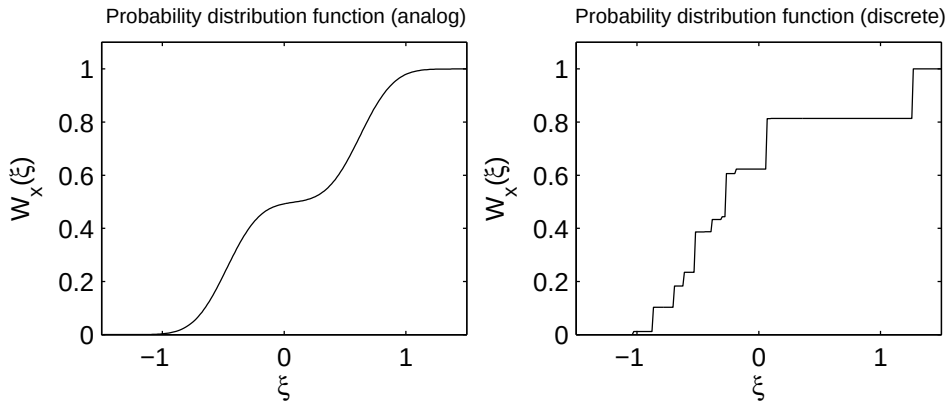


Figure 8.1. Examples of two different probability distribution functions.

The multi-dimensional probability distribution function of several random variables is defined in a similar way. For two random variables X and Y, the two-dimensional distribution is defined as:

$$W_{XY}(\xi_1, \eta_1) = P\{X \leqslant \xi_1, Y \leqslant \eta_1\}. \tag{8.1.4}$$

Another useful statistical characterization of a random variable is the *probability density function*, $w_X(\xi)$, which is the derivative of $W_X(\xi)$:

$$\boxed{w_X(\xi) = \frac{d}{d\xi} W_X(\xi), \qquad W_X(\xi) = \int_{-\infty}^{\xi} w_X(\theta) d\theta} \tag{8.1.5}$$

For the case, in which ξ is defined within the subset of the rational numbers, $w_X(\xi)$ consists exclusively of δ-functions, because $W_X(\xi)$ has a stair-case shape. If ξ is defined over the set of rational numbers, $w_X(\xi)$ is continuous, except for those points, where $W_X(\xi)$ has a discontinuity point. Figure 8.2 shows two examples of probability density functions that correspond to the probability distribution functions shown in Fig. 8.1. For the case, in which $\xi \in \mathbb{R}$ we have the following relations:

$$P\{\xi_1 < X \leqslant \xi_1 + d\xi_1\} = w_X(\xi_1) d\xi_1, \qquad \text{and} \qquad \int_{-\infty}^{\infty} w_X(\xi) d\xi = 1. \tag{8.1.6}$$

Multidimensional probability density functions are defined in a similar manner. Let's consider the two-dimensional case with two random variables X and Y. We have:

$$w_{XY}(\xi; \eta) = \frac{\partial^2 W_{XY}(\xi, \eta)}{\partial \xi \partial \eta}. \tag{8.1.7}$$

In many cases, the time enters as a parameter in the probability distribution- and density functions. If this is important in a given context, then this dependence will be explicitly written, e.g.:

$$w_X(\xi, t).$$

The expected value -the average- of a random variable is given by:

$$E\{X\} = \int_{-\infty}^{\infty} \xi w_X(\xi) d\xi. \tag{8.1.8}$$

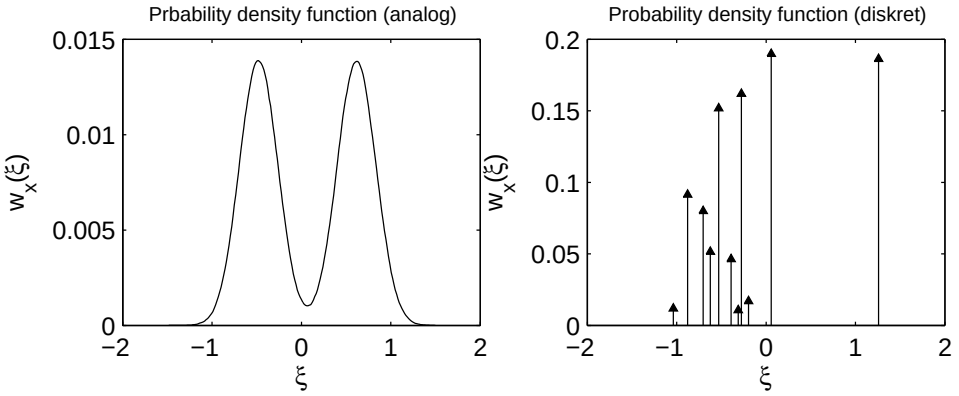


Figure 8.2. Examples of two different probability density functions.

$E\{X\}$ is also referred as the first order moment of X. The *nth moment* of X is defined as:

$$E\{X^n\} = \int_{-\infty}^{\infty} \xi^n w_X(\xi) d\xi. \tag{8.1.9}$$

The *n'th central moment* is given by:

$$E\{(X - E\{X\})^n\} = \int_{-\infty}^{\infty} (\xi - E\{X\})^n w_X(\xi) d\xi. \tag{8.1.10}$$

The second central moment, called also *variance* is:

$$\sigma^2\{X\} = \int_{-\infty}^{\infty} (\xi - E\{X\})^2 w_X(\xi) d\xi. \tag{8.1.11}$$

If X and Y are two random variables, then:

$$E\{XY\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi\eta w_{XY}(\xi, \eta) d\xi d\eta \tag{8.1.12}$$

The central moment

$$E\{(X - E\{X\})(Y - E\{Y\})\} \tag{8.1.13}$$

is called also *covariance* of X and Y.

If k, q, and a are constants, it follows that:

$$\begin{aligned} E\{kX\} &= kE\{X\} \text{ and } E\{kX + qY\} = kE\{X\} + qE\{Y\} \\ E\{X + a\} &= E\{X\} + a \\ \sigma^2\{kX + a\} &= k^2\sigma^2\{X\}. \end{aligned} \tag{8.1.14}$$

Furthermore, for the variance we have:

$$\sigma^2\{X\} = E\{(X - E\{X\})^2\} = E\{X^2\} - E^2\{X\}. \tag{8.1.15}$$

The covariance is given by:

$$E\{(X - E\{X\})(Y - E\{Y\})\} = E\{X\}E\{Y\} - E\{X\}E\{Y\}. \tag{8.1.16}$$

If V is a complex random variable, i.e. $V = X + jY$, then $E\{V\} = E\{X\} + jE\{Y\}$. The variance of V is:

$$\sigma^2\{V\} = E\{|V - E\{V\}|^2\}. \tag{8.1.17}$$

Similarly, the covariance of two complex random variables V and U is:

$$E\{(V - E\{V\})(U^* - E\{U^*\})\}. \tag{8.1.18}$$

Distribution	ξ-region	$w_x(\xi)$	$E\{X\}$	$\sigma^2\{X\}$
Rectangular	$c < \xi < d$	$\frac{1}{d-c}$	$\frac{d+c}{2}$	$\frac{(d-c)^2}{12}$
Normal (Gauss)	$-\infty < \xi < +\infty$	$\frac{1}{\sqrt{2\pi s^2}} \exp\left(-\frac{(\xi-m)^2}{2s^2}\right)$	m	s^2
Rayleigh	$0 < \xi < +\infty$	$\frac{\xi}{s^2} \exp\left(-\frac{\xi^2}{2s^2}\right)$	$s\sqrt{\frac{\pi}{2}}$	$(2-\frac{\pi}{2})s^2$
$\chi^2 \quad (N=1)$	$0 < \xi < +\infty$	$\frac{1}{\sqrt{2\pi s^2}} \frac{1}{\sqrt{\xi}} \exp\left(-\frac{\xi}{2s^2}\right)$	s^2	$2s^4$
χ^2	$0 < \xi < +\infty$	$\frac{1}{2^{\frac{N}{2}} s^N \Gamma(\frac{N}{2})} \xi^{\frac{N-2}{2}} \exp\left(-\frac{\xi}{2s^2}\right)$	Ns^2	$2Ns^4$
2D Gauss	$-\infty < \xi, \eta < +\infty$	$\frac{1}{2\pi s_1 s_2 \sqrt{1-r^2}} \times \exp\left[-\frac{1}{2(1-r^2)}\left(\frac{(\xi-m_1)^2}{s_1^2} + \frac{(\eta-m_2)^2}{s_2^2} - \frac{2r(\xi-m_1)(\eta-m_2)}{s_1 s_2}\right]\right)$		
	Parametre	$m_1, m_2, s_1, s_2, r \quad (\lvert r\rvert < 1)$		

Table 8.1. A table of probability distribution functions.

8.1.2 Relations between random variables

Two random variables X and Y are said to be *independent* if

$$W_{XY}(\xi,\eta) = W_X(\xi)W_Y(\eta), \qquad \text{or} \qquad w_{XY}(\xi,\eta) = w_X(\xi)w_Y(\eta) \tag{8.1.19}$$

This definition can be extended to more random variables.

Two random variables X and Y are said to be *uncorrelated* if the mean of their product is equal to the product of their means:

$$E\{XY\} = E\{X\}E\{Y\}. \tag{8.1.20}$$

Two variables are *orthogonal* if the mean of their product is 0:

$$E\{XY\} = 0. \tag{8.1.21}$$

It can be seen that independent random variables are also uncorrelated. The opposite is not always true. Let's consider two variables X and Y, which both have normal distribution (see Table 8.1). If these two variables are not correlated, i.e. $r = 0$, then they are also independent.

If a random variable U is the *sum of two random variables* X and Y, i.e.:

$$U = X + Y \tag{8.1.22}$$

we can find the probability density function of U from the two-dimensional probability density function of X and Y by:

$$w_U(\xi) = \int_{-\infty}^{\infty} w_{XY}(\xi-\eta,\eta)d\eta. \tag{8.1.23}$$

If X and Y are independent, then this expression is reduced to

$$w_U(\xi) = \int_{-\infty}^{\infty} w_X(\eta)w_Y(\xi-\eta)d\eta \tag{8.1.24}$$

or

$$w_U(\xi) = w_X(\xi) * w_Y(\xi). \tag{8.1.25}$$

These equations can be extended to an arbitrary number of random variables.

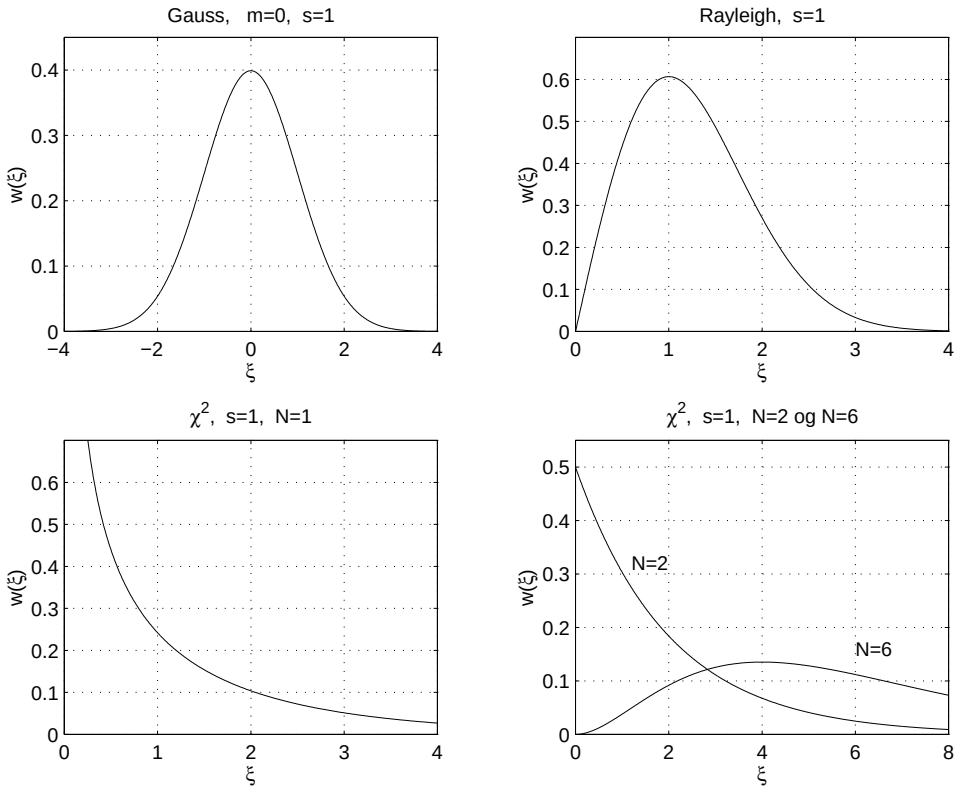


Figure 8.3. Examples of different probability density functions.

8.1.3 Function of a random variable

Often, we would like to work not with the original random variable, but with another random variable that has been derived from the original (the result of processing a random signal). The conversion can be made either through a look-up table, or through one or more mathematical manipulations.

In such cases, we want to be able to compute the probability density function of the new variables, as well as their expected values and other moments.

Let's consider the random variable X with a probability density function $w_X(\xi)$, and the variable Y which is a function of X:

$$Y = \beta(X), \tag{8.1.26}$$

where $\eta = \beta(\xi)$ is the desired conversion of variable. The probability density function $w_Y(\eta)$ can be calculated through an appropriate summation of probabilities.

If $w_X(\xi)$ consists exclusively of δ-functions, then placement and strength of these δ-functions can be found easily through the mapping $\eta = \beta(\xi)$.

If $w_X(\xi)$ does not contain δ-functions, and the function $\eta = \beta(\xi)$ is differentiable, then $w_Y(\eta)$ can be found from the expression:

$$P\{Y \in [\eta, \eta + d\eta]\} = \sum_q P\{X \in I_q\}, \tag{8.1.27}$$

where I_q denotes those intervals, where X can be found, given that Y is in the desired interval of values (in the case of surjections there may be several values of ξ which map to a given η). The magnitude of the probability $P\{X \in I_q\}$ for one of

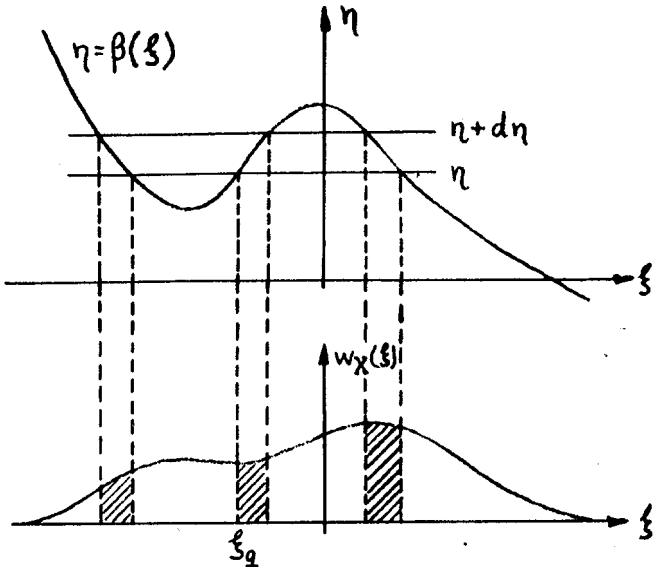


Figure 8.4. Mapping of the probability density function.

the intervals within the sum is (see Fig. 8.4):

$$w_X(\xi_q)\frac{1}{|\beta'(\xi_q)|}, \tag{8.1.28}$$

where ξ_q are the roots of the equation $\beta(\xi) - \eta = 0$, and

$$\beta'(\xi) = \frac{d\beta(\xi)}{d\xi}. \tag{8.1.29}$$

If $\beta'(\xi)$ is a zero only for a finite number of values of ξ, then the inverse function $\xi = \gamma(\eta)$ of $\eta = \beta(\xi)$ exists in these intervals, and we get:

$$w_Y(\eta) = \sum_q \frac{w_X(\xi)}{|\beta'(\xi)|} = \sum_q \frac{w_X(\gamma(\eta))}{|\beta'(\gamma(\eta))|}. \tag{8.1.30}$$

In some applications the function $\eta = \beta(\xi)$ can be a constant in certain intervals along the ξ-axis. In such cases $w_Y(\eta)$ will contain δ-functions, whose strength can be found by integrating $w_X(\xi)$ over the respective intervals.

When $w_X(\xi)$ is of complex character, then it can be treated by combining the above mentioned two methods.

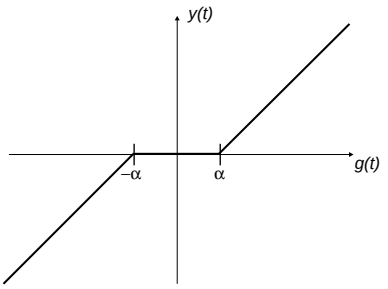
It can be shown that if $Y = \beta(X)$, then:

$$E\{Y^q\} = E\{\beta^q(X)\} \int_{-\infty}^{\infty} \beta^q(\xi) w_X(\xi) d\xi \tag{8.1.31}$$

Example 8.1 (Amplifier with a non-linear transfer characteristics)
A power amplifier of audio signals has a transfer characteristics given by:

$$y(t) = \begin{cases} k(g(t) + \alpha) & \text{for } g(t) < -\alpha \\ k(g(t) - \alpha) & \text{for } g(t) > \alpha \\ 0 & \text{for } |g(t)| \leqslant \alpha \end{cases} \tag{8.1.32}$$

Here, $g(t)$ is the input signal to the amplifier, $y(t)$ is the corresponding output signal, and α is a positive real constant. The transfer characteristic is shown below:



A noise source $x(t)$ is connected to the input of the amplifier. The probability density function of $x(t)$ is defined as:

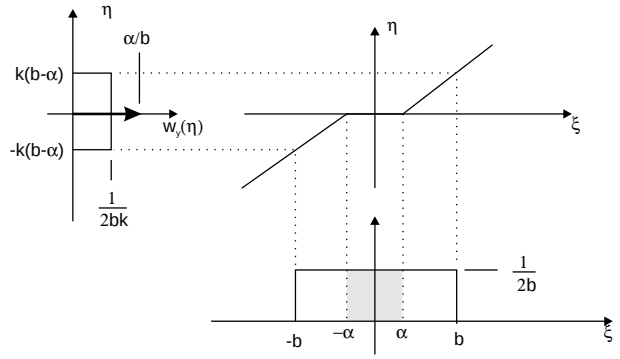
$$w_x(\xi) = \begin{cases} \beta & \text{for } |\xi| \leqslant b \\ 0 & \text{otherwise} \end{cases} \tag{8.1.33}$$

Find β and the probability density function of the output signal $y(t)$ expressed via b, α, and k, when $b \leqslant \alpha$ and $b > alpha$. Calculate the power of the output signal for both cases.

The probability density functions:

Since $\int_{-b}^{b} w_x(\xi)d\xi = 1$, we get $\beta = \frac{1}{2b}$.

For the output signal when $b > \alpha$ we get:



$w_y(\eta)$ consists of a delta function and rectangular density. The amplitude of the delta function is:

$$\frac{1}{2b}2\alpha = \frac{\alpha}{b}$$

The height of rectangular density function is found as:

$$\beta(\xi) = k(\xi - \alpha)$$

$$\beta'(\xi) = k$$

$$w_y(\eta) = \sum_q \frac{w_x(\xi)}{|\beta'(\xi)|} = \frac{\frac{1}{2b}}{|k|} = \frac{1}{2bk}$$

When $b < \alpha$ the output is always 0, and the probability density function is just a delta function with amplitude of 1. The power of the noise is 0. When $b > \alpha$ the power is:

$$\begin{aligned} P_y &= \int_{-k(b-\alpha)}^{+k(b-\alpha)} \eta^2 \left(\frac{1}{2bk} + \frac{\alpha}{b}\delta(\eta) \right) d\eta \\ &= \frac{1}{2bk}\frac{1}{3}((k(b-\alpha))^3 + (k(b-\alpha))^3) \\ &= \frac{k^3(b-a)^3}{3bk} = \frac{k^2(b-a)^3}{3b} \end{aligned}$$

8.2 Random Process

In this section we develop the theory behind the characterization of random processes. After the terms are defined we look into the class of stationary and ergodic random processes. The section finishes with the treatment of Gaussian random processes, known also as normal random process.

8.2.1 Definitions

The word process (lat. movement) denotes a sequence of changes of the properties of a system or object. These may appear deterministic or stochastic in nature to the observer. Just as a random variable may be thought of as mapping from the sample space of a given experiment into the set of real or complex numbers, a *random process* represents mapping of the sample space into a set of signals. Thus a random processes is a collection or *ensemble* of signals. In the following we will use the capital letters to denote random processes:

$$X(t), Y(t) \quad \text{or} \quad X(n), Y(n) \tag{8.2.1}$$

for analog and digital signals, respectively. Another term for a random process is *stochastic* process and in the rest of the chapter these will be used interchangeably.

The process must be considered at a fixed time instance, to apply the mathematical apparatus used to treat random variables. In this way we have the values of the signals for a given time instance, which random values are tied to the outcomes of the experiment. In other words, the values of the signal at the time instance $t = t_0$ represent stochastic variables, which can be characterized by the appropriate probability density functions. The simplest of these are one-dimensional of the type:

$$w_X(\xi; t_0) \qquad \text{or} \qquad w_X(\xi; n_0). \tag{8.2.2}$$

The expected value or the *mean* of a random process is also a function of time and can be obtained by:

$$E\{X(t)\} = \int_{-\infty}^{\infty} \xi w_X(\xi; t) d\xi = \mu_1(t) \tag{8.2.3}$$

Similarly, the qth moment, calculated as

$$E\{X^q(n)\} = \int_{-\infty}^{\infty} \xi^q w_X(\xi; n) d\xi = \mu_q(n) \tag{8.2.4}$$

will also be time dependent.

A process is linked to signals, not to numbers that indicate the outcome of a given experiment. Therefore a random process is not described in its entireness by giving $w_x(\xi; t)$. A complete description would usually require the knowledge of the properties of the process at different times t_1, t_2, t_3, ... Such description is often impossible to handle in practice.

If a process is considered at two time instances, t_1 and t_2, we get a pair of numbers, whose elements each in its own right is a random variable. The two-dimensional probability density function $w_X(\xi_1, \xi_2; t_1, t_2)$ can be used to describe the behavior of this pair of numbers. It is used in the calculations of the different moments (expected values), which could be of interest. These expected values will in general are a function of both parameters t_1 and t_2.

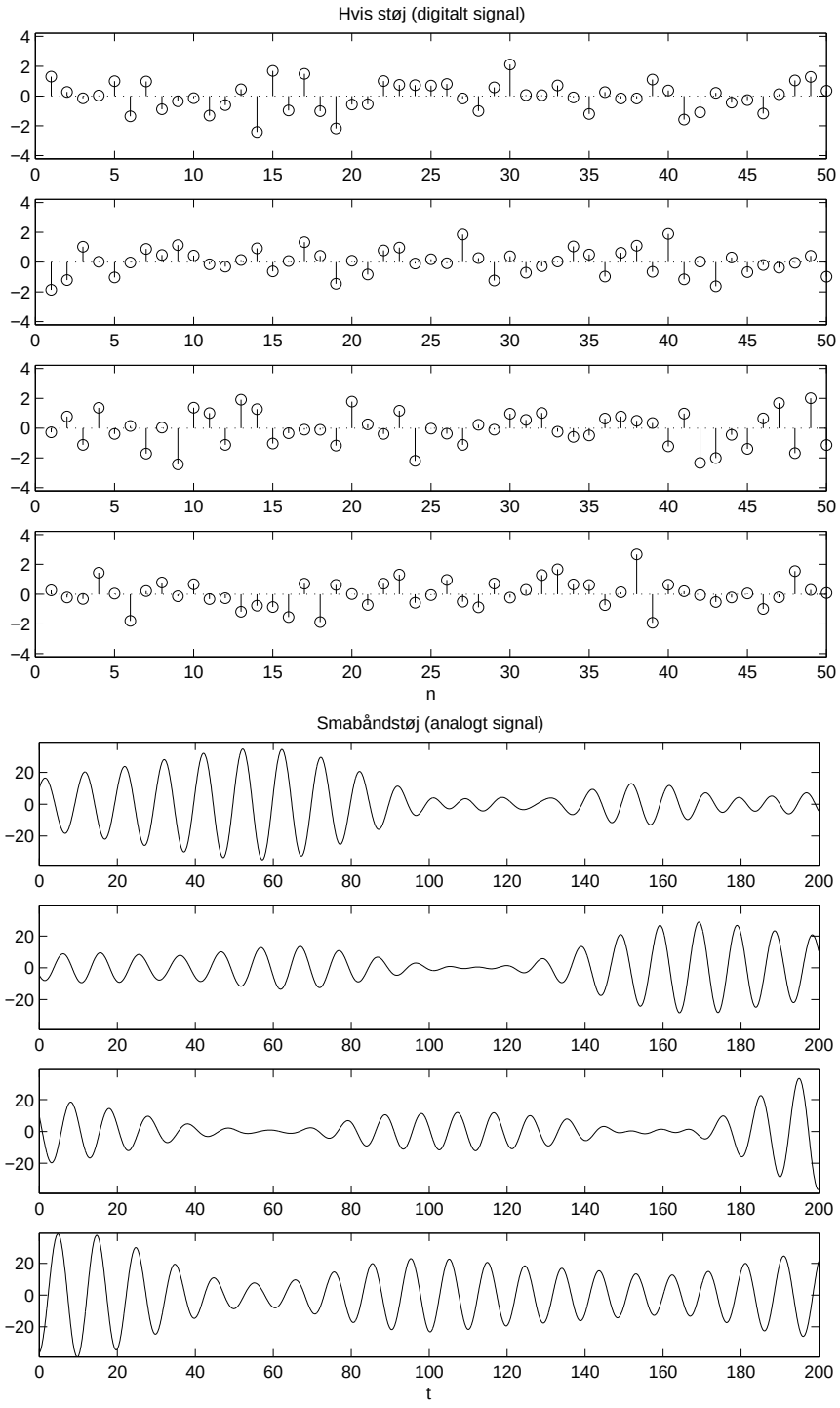


Figure 8.5. Examples of different realizations of analog and digital random signals.

The expected value $E\{X(t_1)X(t_2)\}$ plays an essential part in the characterization of the process. It is known as the *autocorrelation* of the signal and is calculated as:

$$\boxed{R_X(t_1,t_2) = E\{X(t_1)X(t_2)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \xi_1\xi_2 w_X(\xi_1,\xi_2;t_1,t_2)d\xi_1 d\xi_2} \qquad (8.2.5)$$

In the following, the autocorrelation will be denoted as $R_X(t_1, t_2)$ or just $R(t_1, t_2)$. The corresponding expression for digital signals is:

$$R_X(n_1, n_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi_1 \xi_2 w_X(\xi_1, \xi_2; n_1, n_2) d\xi_1 d\xi_2 \tag{8.2.6}$$

Notice that the autocorrelation function of a discrete-time signal is also discrete in time.

A further increase in the number of time instances used in the description of signals is straightforward. The practical problems accompanying such approach, however, do not justify its use. This is why, people usually stop at the autocorrelation and accept the hereby following incompleteness of the signal description.

8.2.2 Stationary random process

If the probability density functions (of arbitrary high order) of a stochastic process are independent of time, then the process is said to be *stationary in the strict sense*. Mathematically, this condition is expressed as:

$$w_X(\xi; t) = w_X(\xi). \tag{8.2.7}$$

As said, a stationary process in the strict sense will have qth order density function independent of time, and the qth-order moment will also be independent of time:

$$E\{X^q(t)\} = \int_{-\infty}^{\infty} \xi^q w_X(\xi) d\xi, \qquad q \quad \text{integer.} \tag{8.2.8}$$

Furthermore, the joint probability density function of a stationary process

$$w_X(\xi_1, \xi_2; t_1, t_2) = w_X(\xi_1, \xi_2; |t_1 - t_2|) \tag{8.2.9}$$

will depend only on the difference between t_1 and t_2. This difference is often denoted by τ, and the autocorrelation of the stationary process becomes:

$$\boxed{\begin{aligned} R_X(\tau) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi_1 \xi_2 w_X(\xi_1, \xi_2; |\tau|) d\xi_1 d\xi_2 \\ R_X(k) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi_1 \xi_2 w_X(\xi_1, \xi_2; |k|) d\xi_1 d\xi_2, \end{aligned}} \tag{8.2.10}$$

for analog and digital signals, respectively.

Continuing to a higher-order joint density functions, a process is said to be *stationary of order p* if pth-order probability density functions are independent of time.

As said, we will be interested only in the mean and the autocorrelation of a process. If these are time-independent, we get a *wide-sense stationary* (WSS) process, which is defined as follows.

Wide Sense Stationarity: A random process $x(n)$ is said to be wide-sense stationary if:

1. The mean of the process is constant $E\{X(t)\} = \text{const.}$
2. The auto-correlation function depends only on the difference in time $R_X(t_1, t_2) = R_X(|t_1 - t_2|)$.
3. The variance of the process is finite $\sigma_X^2 < \infty$.

8.2.3 Ergodic random process

In many applications it is necessary to find the mean and the autocorrelation of a random process. As it was shown in the previous section, these measures require that we have an *ensemble* of realizations of the random process, which is often impossible. A certain class of processes, known as ergodic random processes occupy a special place among random processes. An *ergodic process* is a process generating uniform signals, that is signals which share the same statistical properties. In these signals it is possible to substitute the *ensemle* averages with averages in *time*. So having any of these signals is sufficient to characterize the process completely.

The expected values of the ergodic signals (which require knowledge of the statistical properties of the whole ensemble of signals) are identical to the time averages of the signal (which require knowledge of only a single of the signals of the ensemble).

We will use the notation $\langle\cdot\rangle$ to denote a time-average of the signal(s), which are inside the angle brackets. The time averages of an analog $x(t)$ and digital $x(n)$ signals are given by:

$$\boxed{\begin{aligned}\langle x(t)\rangle &= \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T} x(t)dt\\ \langle x(n)\rangle &= \lim_{N\to\infty}\frac{1}{N}\sum_{N} x(n)\end{aligned}} \tag{8.2.11}$$

where $\sum_N$ means that the summation spans over N signal samples, chosen in such a way, that through the limit transition one encompasses the whole signal.

The fact that a stationary random process process is ergodic means that the mean values in time of any of the signals generated by the process are equal to the mean values calculated over the ensemble of realizations. For example:

$$\begin{aligned}E\{X(t)\} &= \langle x(t)\rangle\\ E\{X(n)X(n+k)\} &= \langle x(n)x(n+k)\rangle\end{aligned} \tag{8.2.13}$$

Example 8.2 (Example of a stochastic process)
Let's consider the following experiment. We have a box of resistors of the same type (e.g. 150 $\Omega \pm 10\%$). Typically these resistor will have different *actual* resistance, since there is a variation in the production process. Let's pick randomly a number of resistors, and apply voltage on them. These resistors will create electric noise signal due to the thermal movement of electrons. The measured noise signals is that signal, which is assigned to the individual outcomes of the experiment. Let's further assume that we like fresh air, and we conduct the experiment outdoors, where the ambient temperature varies.

Since we have assumed that the resistors are of the "same type", we have generated a stochastic process $X(t)$.

The stochastic process is not stationary, because the probability density function

$$w_X(\xi;t) \tag{8.2.14}$$

of the process will have parameters which vary with the ambient temperature, in other words - the time.

If we subject the resistors to completely identical conditions (temperature, moisture, etc), the process will become stationary. The process is, however, not ergodic. Let's assume for the sake of simplicity that:

$$E\{X(t)\} = 0, \tag{8.2.15}$$

which means that there is no DC component of the current flowing through the resistors. If we calculate the power of the resulting signal we will find out that:

$$\langle x_p^2(t)\rangle \neq \langle x_q^2(t)\rangle, \tag{8.2.16}$$

where $x_p(t)$ and $x_q(t)$ are two randomly chosen signals from the constructed process. They are different, because the actual resistance of the resistors varies from resistor to resistor.

In other words, it is not possible to characterize the process based on the study of only one of the signals assigned to the outcomes of the experiment.

Now, if the process is modified such as the resistors in the experiment are fully identical, and are kept under the same conditions, then the process becomes ergodic. Under these conditions we will have:

$$\langle x_p^2(t)\rangle = \langle x_q^2(t)\rangle, \text{ for all } p \text{ and } q. \tag{8.2.17}$$

For the sake of clarity we must emphasize that this is not the same as

$$x_p(t) \equiv x_q(t) \text{ for } p \neq q, \tag{8.2.18}$$

because the signals emanate from two different resistors.

8.2.4 Stationary, normal random process

The so-called normal distribution plays an important role in nature. It is even part of the plot of the novel Jurassic Park by Michael Crichton. The people who work at the park are thrilled because the histogram shows a nice normal distribution, which they say is what you'd expect from height data from a typical, healthy, biological population. Many other processes can be modeled well using a normal distribution, and this is the reason why it is treated in this section.

The one-dimensional probability density function of a normal random process is given by[1]

$$\boxed{w_X(\xi) = \frac{1}{\sqrt{2\pi \cdot s^2}} \cdot \exp\left(-\frac{(\xi - m)^2}{2s^2}\right), \qquad -\infty < \xi < \infty} \tag{8.2.19}$$

(see also table 8.1). The constants m and s are given by:

$$\begin{aligned} m &= E\{X(t)\} \\ s^2 &= E\{(X(t) - E\{X(t)\})^2\} = E\{X^2(t)\} - E^2\{X(t)\}. \end{aligned} \tag{8.2.20}$$

If the mean value of the process is 0, $E\{X(t)\} = 0$, then

$$s^2 = E\{X^2(t)\} = R(0), \tag{8.2.21}$$

where $R(\tau)$ is the autocorrelation function of the process.

It can be shown that the higher-order even moments of the process are given by:

$$E\{x^{2q}(t)\} = \frac{(2q)!}{2^q q!} s^{2q} \quad , \tag{8.2.22}$$

while all moments of odd order are equal to zero.

The *two-dimensional probability density function* of a stationary normal process with mean value of 0 is:

$$w_X(\xi_1, \xi_2; \tau) = \frac{1}{2\pi\sqrt{R^2(0) - R^2(\tau)}} \cdot \exp\left(-\frac{R(0)\xi_1^2 + R(0)\xi_2^2 - 2R(\tau)\xi_1\xi_2}{2(R^2(0) - R^2(\tau))}\right) \quad . \tag{8.2.23}$$

[1] Notice that strictly speaking, a digital signal cannot be normally distributed.

From this equation it can be seen that if $R(\tau) = 0$ either for discrete values of τ or in intervals of values, then we have:

$$w_X(\xi_1, \xi_2; \tau) = w_X(\xi_1) w_X(\xi_2) \quad . \tag{8.2.24}$$

In other words, the values of the random signal which are τ seconds apart are independent under these conditions.

It is possible to show that all multidimensional probability density functions are functions of $R(\tau)$ (and m, if $m \neq 0$).

8.3 Correlation functions and power density spectrum

As mentioned in Section 8.2, the first and the second order moments are important means to characterize random processes in practice. The mean of a process is an indication of whether there is a DC (direct current) component of the signal, and the autocorrelation function, which is a function of time, is used as a basis for the spectral description of random processes.

8.3.1 Autocorrelation functions

The *auto correlation functions* of the stationary signals $X(t)$ and $X(n)$ are defined as:

$$R_X(\tau) = E\{X(t)X(t+\tau)\} \quad \text{and} \quad R_X(k) = E\{X(n)X(n+k)\} \tag{8.3.1}$$

They express the average dependence (relation) between the values of the signal that are τ seconds or k samples apart.

Closely related to the autocorrelation functions is the *autocovariance*:

$$C_X(\tau) = E\{[X(t) - E\{X(t)\}][X(t+\tau) - E\{X(t\}]\} \quad . \tag{8.3.2}$$

The auto-covariance becomes identical to the autocorrelation, when there is no DC component present in the signal, i.e. $E\{X(t)\} = 0$. For discrete-time random signals, the auto-covariance is defined in a similar way.

The autocorrelation functions $R_X(\tau)$ and $R_X(k)$ are even functions because the two-dimensional probability density functions $w_X(\xi_1, \xi_2; \tau)$ and $w_X(\xi_1, \xi_2; k)$ depend only on the value of the time parameter. In other words:

$$R_X(\tau) = R_X(-\tau) \qquad \text{and} \qquad R_X(k) = R_X(-k). \tag{8.3.3}$$

Analogous to the relations that exist for random variables, the *variance* of a discrete-time signal $x(n)$ is defined as:

$$\sigma^2\{X(n)\} = E\{[X(n) - E\{X(n)\}]^2\}. \tag{8.3.4}$$

The variance of an analog signal is defined in a similar way. Using the fact that

$$\int_{-\infty}^{\infty} (\xi - E\{X(n)\})^2 w_X(\xi) d\xi = E\{X^2(n)\} - E^2\{X(n)\} \tag{8.3.5}$$

it can be shown that

$$R_X(0) = E\{X^2(n)\} = \sigma^2\{X(n)\} + E^2\{X(n)\}. \tag{8.3.6}$$

Hence the autocorrelation function at lag 0 is always non-negative:

$$R_X(0) \geqslant 0 \tag{8.3.7}$$

Consider the random signal $Y(t)$ constructed from an arbitrary stationary random signal $X(t)$ as:

$$Y(t) = X(t) \pm X(t+\tau) \tag{8.3.8}$$

The autocorrelation function of $Y(t)$ at lag zero is

$$R_Y(0) = E\{(X(t) \pm X(t+\tau))^2\} = 2R_X(0) \pm 2R_X(\tau). \tag{8.3.9}$$

Since $R_Y(0) \geqslant 0$ it follows that:

$$\boxed{R_X(0) \geqslant |R_X(\tau)| \qquad \text{for} \qquad \tau \neq 0} \tag{8.3.10}$$

The autocorrelation functions of most random signals approaches a positive constant at infinite time lags:

$$\lim_{|\tau|\to\infty} R(\tau) = R_\infty \qquad \text{and} \qquad \lim_{|k|\to\infty} R(k) = R_\infty, \tag{8.3.11}$$

where R_∞ is a positive constant. It can be shown that

$$\boxed{R_\infty = E^2\{X(t)\} \qquad \text{and} \qquad R_\infty = E^2\{X(n)\}} \tag{8.3.12}$$

This limit value is in other words the DC-power of the signal[2]. Hence the AC-power of the random process must be given by:

$$\sigma^2\{X(t)\} \qquad \text{and} \qquad \sigma^2\{X(n)\}, \tag{8.3.13}$$

since the *total power* P is given by the value of the autocorrelation function at time lag 0.

As mentioned previously, the time-average values can be used instead of the expected values. In other words, the power of a signal is

$$P = \langle x_q^2(t) \rangle \qquad \text{and} \qquad P = \langle x_q^2(n) \rangle. \tag{8.3.14}$$

8.3.2 Spectrum of a random signal

The spectrum of a random signal is defined as the Fourier transform of the autocorrelation function of the signal. For analog signals this means that:

$$R_X(\tau) \leftrightarrow S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau. \tag{8.3.15}$$

$R_X(\tau)$ is an even function, hence the $S_X(f)$ is real and even function of the frequency f. Using the fact that the power P is $P = R_X(0)$ and that $e^{-j\pi f 0} \equiv 1$, we have

$$P = R_X(0) = \int_{-\infty}^{\infty} S_X(f) df \tag{8.3.16}$$

In other words, it can be seen that $S_X(f)$ is a *power density spectrum*. It tells us in which frequencies the random signal has spectral components. It can also be shown that:

$$S_X(f) \geqslant 0. \tag{8.3.17}$$

Similarly the spectrum of a random discrete-time signal is given as:

$$R_X(k) \overset{\Delta T}{\leftrightarrow} S_X(f) = \sum_{k=-\infty}^{\infty} R_X(k) e^{-j2\pi f k \Delta T}. \tag{8.3.18}$$

[2]You can imagine that the signal is either voltage applied on or current flowing through a 1Ω resistor. The power is that one dissipated by the resistor.

$S_X(f)$ is an even function of f in this case too. It is also periodic with a a period of $\frac{1}{\Delta T}$. For discrete-time signals, $S_X(f)$ is a *power spectrum*. Often the terms power-density spectrum and power spectrum are used indiscriminately.

The total effect of a discrete-time random signal is found as:

$$P = R_X(0) = \frac{1}{f_s} \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} S_X(f)df, \tag{8.3.19}$$

where $f_s = \frac{1}{\Delta T}$ is the sampling frequency.

As it can be seen, the spectral characteristics of random signals are characterized by using a deterministic signal with finite energy - $R_X(\tau)$ or $R_X(k)$.

8.3.3 Cross correlation functions

In many practical applications we have random signals which are a combination of random signals with known properties. To be able to characterize such signals through their autocorrelation function, we need to introduce the terms cross-correlation and cross spectrum.

The *cross correlation* of two stationary random analog signals $X(t)$ and $Y(t)$ is defined as:

$$R_{XY}(\tau) = E\{X(t), Y(t+\tau)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi\eta w_{XY}(\xi, \eta; \tau)d\xi d\eta. \tag{8.3.20}$$

The cross correlation function expresses the level of connection (coherence) between two signals $X(t)$ and $Y(t)$.

The corresponding expression for discrete-time signals is:

$$R_{XY}(k) = E\{X(n)Y(n+k)\} \tag{8.3.21}$$

The probability density functions

$$w_{XY}(\xi, \eta; \tau) \qquad \text{and} \qquad w_{XY}(\xi, \eta; k)$$

are usually not even functions of τ or k. Hence, the cross correlation functions are not even too:

$$R_{XY}(\tau) \neq R_{XY}(-\tau) \quad (\tau \neq 0) \qquad R_{XY}(k) \neq R_{XY}(-k) \quad (k \neq 0). \tag{8.3.22}$$

On the other hand it can be shown through a simple substitution of variables that:

$$R_{XY}(\tau) = R_{YX}(-\tau) \quad \text{and} \quad R_{XY}(k) = R_{YX}(-k). \tag{8.3.23}$$

Consider the equation:

$$E\{(X(t) + a(Y(t+\tau))^2)\} = R_X(0) + a^2 R_Y(0) + 2aR_{XY}(\tau) \geqslant 0 \tag{8.3.24}$$

where a is a real constant and $X(t)$ and $Y(t)$ are analog random signals. This equation is a square equation of a. It is a *convex* function of a. In order to be valid for all values of a, it must not have any solutions.Therefore, the following condition must be fulfilled:

$$(2R_{XY}(\tau))^2 - 4R_Y(0)R_X(0) < 0, \tag{8.3.25}$$

hence

$$\boxed{R_{XY}^2 < R_X(0)R_Y(0)} \tag{8.3.26}$$

In the special case when $a = 1$ or $a = -1$ we get

$$\boxed{2|R_{XY}(\tau)| < R_X(0)R_Y(0)} \tag{8.3.27}$$

The corresponding expressions for discrete-time signals are:

$$R_{XY}^2(k) < R_X(0) + R_Y(0) \tag{8.3.28}$$

and

$$2|R_{XY}(k)| < R_X(0) + R_Y(0) \tag{8.3.29}$$

Two random signals $X(n)$ and $Y(n)$ are said to be uncorrelated *for a given time* lag $k = k_1$ if

$$R_{XY}(k_1) = E\{X(n)Y(n-k_1)\} = E\{X(n)\}E\{Y(n)\} \tag{8.3.30}$$

If the last expression is valid for *any* value of k, i.e.

$$R_{XY}(k) = E\{X(n)\}E\{Y(n)\}, \tag{8.3.31}$$

then the two signals are *uncorrelated*.

$X(n)$ and $Y(n)$ are said to be *orthogonal signals* if

$$R_{XY}(k_1) = 0 \quad \text{and} \quad R_{XY}(k) = 0 \quad (\text{for all } k) \tag{8.3.32}$$

Notice that if only one of the signals has a zero DC component, then the two terms *uncorrelated* and *orthogonal* become identical.

While the above conditions impose some restrictions on the expected values of the signals, two signals are said to be *statistically independent* if their multidimensional distributions (and pdf) can be written as a product of the distributions (and pdf):

$$w_{XY}(\xi, \eta; k) = w_X(\xi) \cdot w_Y(\eta). \tag{8.3.33}$$

Consequently, statistically independent signals are also uncorrelated. The converse is valid only for signals, whose two-dimensional probability density functions have normal distribution.

The above is valid for analog signals too. If the signals are ergodic, then the expected values can be substituted by time averages.

8.3.4 Cross spectra of two random signals

The cross-spectrum of random signals is defined as the Fourier transform of the cross-correlation function of the two signals:

$$R_{XY}(\tau) \leftrightarrow S_{XY}(f) \quad \text{and} \quad R_{XY}(k) \overset{\Delta T}{\leftrightarrow} S_{XY}(f). \tag{8.3.34}$$

It must be noted that $S_{XY}(f)$ usually is complex and that

$$S_{XY}(f) = S_{XY}^*(-f) \tag{8.3.35}$$

and

$$S_{YX}(f) = S_{XY}^*(f). \tag{8.3.36}$$

The cross-spectrum carries information about within which parts of the spectrum the two signals has some resemblance. In other words, the values of the cross-spectrum in that frequency range must be $\neq 0$.

8.4 Linear and non-linear combination of random signals

8.4.1 Multiplication of a random signal by a constant

The multiplication of a random signal by a constant is expressed as:

$$\boxed{Y(t) = aX(t)} \tag{8.4.1}$$

where a is a real constant which is $\neq 0$, and $E\{X(t)\} = a$

The one-dimensional probability density function of the signal $Y(t)$ can be found as described in section 8.1.3:

$$w_Y(\eta) = \frac{1}{|a|} w_X(\eta/a) \tag{8.4.2}$$

The power P_Y of the signal $Y(t)$ is easily found as

$$P_Y = a^2 P_X, \tag{8.4.3}$$

where P_X is the power of $X(t)$.

From the expression

$$R_Y(\tau) = E\{aX(t)aX(t+\tau)\} = a^2 R_X(\tau) \tag{8.4.4}$$

it follows that

$$S_Y(f) = a^2 S_X(f). \tag{8.4.5}$$

It must be noted that if $X(t)$ is stationary then $Y(t)$ is stationary too.

Some applications use a slightly different form of signal combination:

$$\boxed{U(t) = aX(t) + b} \tag{8.4.6}$$

where a and b are real constants and $a \neq 0$.

The above expression means that the signal $aX(t)$ is modified by a constant (DC) signal b. Repeating the considerations from above we get:

$$w_U(\eta) = \frac{1}{|a|} w_x((\eta - b)/a). \tag{8.4.7}$$

The power of $U(t)$ is (given that $E\{X(t)\} = 0$)

$$P_U = a^2 P_X + b^2. \tag{8.4.8}$$

The autocorrelation function of $U(t)$ is

$$R_U(\tau) = a^2 R_X(\tau) + b^2. \tag{8.4.9}$$

The power density spectrum is

$$S_U(f) = a^2 S_X(f) + b^2 \delta(f) \tag{8.4.10}$$

All of these expressions are valid for digital signals too.

8.4.2 Linear combination of two random signals

The linear combination of two random signals $X(t)$ and $Y(t)$ creates a new random signal $U(t)$ which is

$$U(t) = aX(t) + bY(t), \tag{8.4.11}$$

where a and b are real constants. The one-dimensional probability density function of $U(t)$ can be found using the results from section 8.1.3.

The autocorrelation of $U(t)$ is

$$\begin{aligned} R_U(\tau) &= E\{(aX(t) + bY(t))(aX(t+\tau) + bY(t+\tau))\} \\ &= a^2 R_X(\tau) + b^2 R_Y(\tau) + abR_{XY}(\tau) + abR_{YX}(\tau) \end{aligned} \tag{8.4.12}$$

which implies that we must know the autocorrelation functions of $X(t)$ and $Y(t)$, and their cross-correlation to find the autocorrelation of $U(t)$.

The total power of $U(t)$ is

$$P_U = a^2 R_X(0) + b^2 R_Y(0) + 2ab R_{XY}(0) \tag{8.4.13}$$

If $R_{XY}(0)$ is negative and has a certain value, this means that P_U could be less than both P_X and P_Y. The power spectrum of $U(t)$ is

$$S_U(f) = a^2 S_X(f) + b^2 S_Y(f) + 2ab\mathrm{Re}[S_{XY}(f)] \tag{8.4.14}$$

If $X(t)$ and $Y(t)$ are uncorrelated (or statistically independent) and the mean value of one of the signals is 0 (E{X(t)}=0 or E{Y(t)}=0), then the above expressions are simplified as

$$R_U(\tau) = a^2 R_X(\tau) + b^2 R_Y(\tau), \quad P_U = a^2 P_X + b^2 P_Y \tag{8.4.15}$$

and

$$S_U(f) = a^2 S_X(f) + b^2 S_Y(f) \tag{8.4.16}$$

There are similar expressions for discrete-time signals.

8.4.3 Product of two random signals

Some non-stationary signals encountered in practical applications, can be modeled as the product of two random signals $X(t)$ and $Y(t)$, i.e.

$$V(t) = X(t)Y(t) \tag{8.4.17}$$

The result is relatively simple only if the two signals are statistically independent. Its autocorrelation function can be shown to be

$$R_V(\tau) = R_X(\tau)R_Y(\tau). \tag{8.4.18}$$

The same result is valid for digital signals.

8.5 Correlation function for deterministic signals with finite energy

To understand better the filtration of random signals, it is necessary to introduce also the autocorrelation function for signals with finite energy. The autocorrelation $R_h(\tau)$ for the energy signal $h(t)$ is given by:

$$R_h(\tau) = \int_{-\infty}^{\infty} h(t)h(t+\tau)dt \tag{8.5.1}$$

Notice that the integral is not normalized as it in the case of ergodic random signals. An understanding of how the autocorrelation function is created can be obtained from the following equation:

$$R_h(\tau) = h(-\tau) * h(\tau) = \int_{-\infty}^{\infty} h(\theta)h(\theta+\tau)d\theta \tag{8.5.2}$$

It can be seen that the autocorrelation can be found by convolving the signal with a time-reversed copy of itself. This is illustrated in Fig. 8.6

From the autocorrelation function one can find the energy density spectrum:

$$R_h(\tau) = h(-\tau) * h(\tau) \leftrightarrow H^*(f)H(f) = |H(f)|^2 \tag{8.5.3}$$

The corresponding expressions for a finite-energy digital signal are:

$$R_h(k) = \sum_{n=-\infty}^{\infty} h(n)h(n+k) \tag{8.5.4}$$

and

$$R_h(k) = h(-k) * h(k) \overset{\Delta T}{\leftrightarrow} H^*(f)H(f) = |H(f)|^2. \tag{8.5.5}$$

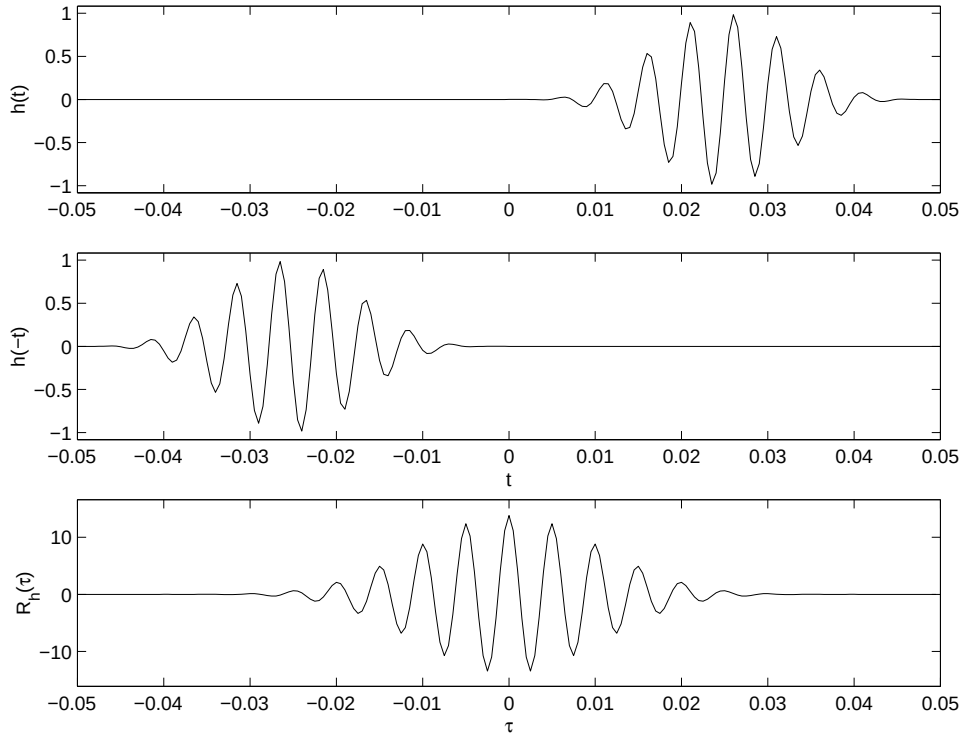


Figure 8.6. The autocorrelation function of a finite energy analog signal.

8.6 Correlation function and power spectrum of periodic signals

Sometimes deterministic periodic signals appear together with random signals. Since the characterization of random signals is based on their autocorrelation functions or their power density spectra, it is useful to define the correlation functions for periodic signals.

8.6.1 Autocorrelation function of periodic signals

The autocorrelation function of an analog periodic signal is defined as (see Fig. 8.7):

$$R_g(\tau) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} g(t)g(t+\tau)dt \tag{8.6.1}$$

$R_g(\tau)$ is again a periodic function with the same period T as the signal $g(t)$. This can easily be verified is τ is replaced by $\tau + T$ in (8.6.1).

The autocorrelation function is also an even function of time τ, i.e. $R_g(\tau) = R_g(-\tau)$. This property can also be verified by substitution of variables. In this case one uses the fact that the integration must not necessarily start at $-\frac{T}{2}$, but must only include an integer number of periods of $g(t)$[3]

From the expression

$$R_g(0) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} g^2(t)dt, \tag{8.6.2}$$

it can be seen that $R_g(0) = P$ is the power of the signals.

[3]Strictly speaking $R_g(\tau) = \lim_{T_1\to\infty} \frac{1}{T_1}\int_{T_1} g(t)g(t+\tau)dt$

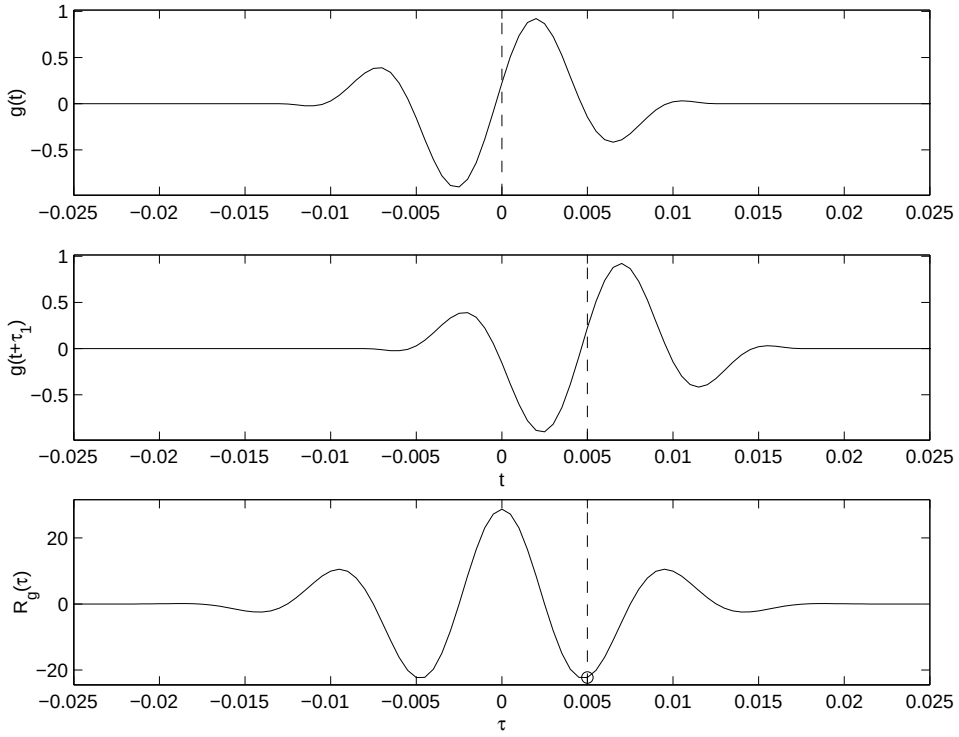


Figure 8.7. Autocorrelation function of a periodic signal.

The autocorrelation of a periodic digital signal $g(n)$ is given by

$$R_g(k) = \frac{1}{N}\sum_{n=0}^{N-1} g(n)g(n+N), \tag{8.6.3}$$

where k is an integer number. Also in this case we have:

$$R_g(k) = R_g(k+N), \qquad R_g(k) = R_g(-k), \qquad \text{and} \qquad P = R_g(0) \tag{8.6.4}$$

8.6.2 Power spectrum of periodic signals

The spectrum $S_p(f)$ of the periodic signals $R_g(\tau)$ is called the power spectrum of the signal $g(t)$:

$$R_g(\tau) \overset{T}{\leftrightarrow} S_g(m) \tag{8.6.5}$$

The reason to use the term power spectrum can easily be demonstrated by rewriting the expression for $R_g(\tau)$ as:

$$R_g(\tau) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} g(-t)g(\tau - t)dt = g(-\tau) * g(\tau). \tag{8.6.6}$$

Hence,

$$S_g(m) = G^*(m)G(m) = |G(m)|^2, \tag{8.6.7}$$

where $g(m) \overset{T}{\leftrightarrow} G(m)$. The total power can be found by

$$P = \sum_{m=-\infty}^{\infty} S_g(m) \tag{8.6.8}$$

Since $R_g(\tau)$ is an even function of τ, then the power spectrum $S_g(m)$ will be a real and even function of m. Notice that $S_g(m)$ is independent of the phase-spectrum of $g(t)$, which implies that different signals can have the same power spectrum and hereby the same autocorrelation function. It can also be seen that $S_g(m) \geqslant 0$. The power spectrum of a periodic digital signal is given as

$$R_g(k) \overset{N}{\leftrightarrow} S_g(m), \tag{8.6.9}$$

and as in the case of periodic analog signals the following relations will be valid:

$$S_g(m) = |G(m)|^2, \qquad P = \sum_{m=0}^{N-1} S_g(m), \qquad \text{and} \qquad S_g(m) \geqslant 0. \tag{8.6.10}$$

8.6.3 Cross-correlation function of two periodic signals

The *cross-correlation function* $R_{gh}(\tau)$ of the periodic analog signals $g(t)$ and $h(t)$ with the same period T is defined by

$$R_{gh}(\tau) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} g(t)h(t+\tau)dt \tag{8.6.11}$$

$R_{gh}(\tau)$ is also periodic in τ with the period T, so that $R_{gh}(\tau) = R_{gh}(\tau + T)$. The cross-correlation function is in general *not* an even function, except if $h(t)$ is proportional to $g(t)$. The following relation is, though, valid

$$\boxed{R_{gh}(\tau) = R_{hg}(-\tau)} \tag{8.6.12}$$

The latter can easily be proven by substituting θ with $\theta = t + \tau$ in the definition of the cross correlation. To show it one must also use the fact the at the integration is carried out over a full period independent of its location in time.

The cross-correlation function for two periodic digital signals $g(n)$ and $h(n)$ is also periodic with the period N as the two signals $g(n)$ and $h(n)$, and is found from:

$$R_{gh}(k) = \frac{1}{N} \sum_{n=0}^{N-1} g(n)h(n+k). \tag{8.6.13}$$

It can easily be seen that $R_{gh}(k) = R_{hg}(-k)$. Notice that although the given expression contain integration over a single period, the following relations are also valid:

$$\lim_{T_1 \to \infty} \frac{1}{T_1} \int_{T_1} g(t)h(t+\tau)dt = R_{gh}(\tau) \tag{8.6.14}$$

and

$$\lim_{N_0 \to \infty} \frac{1}{N_0} \sum_{N_0} g(n)h(n+k) = R_{gh}(k) \tag{8.6.15}$$

8.6.4 Cross power spectrum of two periodic signals

The Fourier transform $S_{gh}(f)$ of $R_{gh}(\tau)$ is called *cross-power spectrum* of the two signals $g(t)$ and $h(t)$. This spectrum is a normal complex function of the complex frequency parameter m. From the expression:

$$R_{gh}(\tau) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} g(-t)h(\tau - t) = h(\tau) * g(-\tau) \tag{8.6.16}$$

it follows that:

$$S_{gh}(m) = H(m)G^*(m) = H(m)G(-m), \tag{8.6.17}$$

where $g(t) \overset{T}{\leftrightarrow} G(m)$ and $h(t) \overset{N}{\leftrightarrow} H(f)$.

The same relations can be proven for digital signals too:

$$R_{gh}(k) \overset{N}{\leftrightarrow} S_{gh}(m) = H(m)G^*(m), \tag{8.6.18}$$

where $g(n) \overset{N}{\leftrightarrow} G(m)$ and $h(n) \overset{N}{\leftrightarrow} H(m)$. The cross power spectrum gives information about those frequencies where both signals have spectral components. It can be seen that there exist many signal pairs which have the same cross-correlation function, hence cross power spectrum.

8.6.5 Correlation functions and power spectra of pure tones

A pure-tone signal, analog or digital is given by

$$g(t) = a_1 \cos(2\pi f_0 t + \theta_1) \tag{8.6.19}$$

and

$$g(n) = a_1 \cos(2\pi \frac{n}{N} + \theta_1), \tag{8.6.20}$$

respectively. In the expressions above a_1 and θ_1 are constants, and the periods are $T = 1/f_0$ and N. These signals have spectral components only at $m = \pm 1$ (only first harmonics). In both cases

$$S_g(\pm 1) = |G(\pm 1)|^2 = \frac{a_1^2}{4} \tag{8.6.21}$$

It follows that the autocorrelation functions for the two signals are:

$$R_g(\tau) = \frac{1}{2} a_1^2 \cos 2\pi f_0 \tau \qquad \text{and} \qquad R_g(k) = \frac{1}{2} a_1^2 \cos 2\pi \frac{k}{N} \tag{8.6.22}$$

Notice that the phase angle of the signals θ_1 is missing in the expressions of the autocorrelation functions. This means that all pure tones with the same frequency and strength have the same autocorrelation function.

The cross-correlation function of two tones with the same frequency can be found again using simple consideration in frequency domain. Let $g(t)$ and $h(n)$ be defined by the expressions above and let $h(t)$ and $h(n)$ be $h(t) = a_2 \cos(2\pi f_0 + \theta_2)$ and $h(n) = a_2 \cos(2\pi n/N + \theta_2)$. In other words, the differences lie in the different amplitude and phase. The cross-power spectrum is:

$$S_{gh}(\pm 1) = \frac{a_1 a_2}{4} e^{\pm j(\theta_2 - \theta_1)}, \tag{8.6.23}$$

which implies that:

$$R_{gh}(\tau) = \frac{a_1 a_2}{2} \cos(2\pi f_0 t + (\theta_2 - \theta_1)) \tag{8.6.24}$$

and

$$R_{gh}(k) = \frac{a_1 a_2}{2} \cos(2\pi \frac{n}{N} + (\theta_2 - \theta_1)). \tag{8.6.25}$$

If the two pure tones do not have the same period, then $R_{gh}(\tau) \equiv 0$ and $R_{gh}(k) \equiv 0$.

8.6.6 Correlation functions of periodic signal with equally large spectral components

If a periodic band-limited signal has spectral components which are real[4] and equally large, then the maximum of this signal is centered at $t = 0$ or $n = 0$, and no other

[4]Remember that real signals with even symmetry hare real spectra.

periodic signal consisting of the same spectral components and having the same power will have values that are larger.

Such a signal ($g(t)$ or $g(n)$) is further characterized by the fact that the autocorrelation function is proportional to the signal itself, i.e. it has the same shape as the signal. This is a direct sequence of the fact that $R_g(\tau) \overset{T}{\leftrightarrow} |G(m)|^2$ and $R_g(m) \overset{N}{\leftrightarrow} |G(m)|^2$.

If the analog signal $h(t)$ has spectral components at the same frequencies as $g(t)$, it follows that the cross-correlation function $R_{gh}(\tau)$ will have the same shape as $h(t)$. It can be shown directly from the relation:

$$S_{gh}(m) = H(m)G^*(m) = k_x H(m), \tag{8.6.26}$$

where k_x is the constant and real value of the spectral components of $g(t)$.

8.6.7 Autocorrelation function of a periodic rectangular signal

The autocorrelation function of the periodic analog signal

$$g(t) = \begin{cases} a & |\, t \,| < \frac{1}{2}\theta \\ 0 & \frac{1}{2}\theta < |\, t \,| < \frac{1}{2}T \end{cases} \tag{8.6.27}$$

where $\theta < \frac{1}{2}T$, is given for positive values of τ by

$$\frac{1}{T}\int_{-\frac{1}{2}\theta}^{\frac{1}{2}\theta-\tau} a^2 dt = a^2\theta(1-\tau/\theta)/T \tag{8.6.28}$$

as shown in Fig. 8.8. For any value of τ, the autocorrelation function is

$$R_g(\tau) = a^2\frac{\theta}{T}\left(1-\frac{|\tau|}{\theta}\right), \qquad |\tau| < \theta. \tag{8.6.29}$$

The same result can be derived for the digital signal

$$g(n) = \begin{cases} a & 0 \leqslant n \leqslant N_0 - 1 \\ 0 & N_0 \leqslant n \leqslant N-1 \end{cases} \tag{8.6.30}$$

where the constant N_0 is $N_0 < \frac{1}{2}N$. For positive values of k one gets the sum

$$\frac{1}{N}\sum_{n=0}^{N_0-1-k} a^2 = \frac{N_0-k}{N}a^2 \tag{8.6.31}$$

and hereby

$$R_g(k) = a^2\frac{N_0}{N}\left(1-\frac{|k|}{N_0}\right), \qquad |k| < N_0 \tag{8.6.32}$$

These results can be also obtained by first finding the power spectrum of $g(t)$ or $g(n)$ and then using the inverse Fourier transform to find the correlation function.

Notice that the above expressions are for signals with a duty cycle below 50 % ($\frac{\theta}{T} < \frac{1}{2}$ and $\frac{N_0}{N} < \frac{1}{2}$). Signals whose duty cycle exceeds 50 % can be treated as the sum of a constant DC signal and a rectangular signal with a duty cycle $< 50\%$, and negative amplitude a.

8.6.8 Finding the spectrum of a signal via cross-correlation

The cross-correlation of a periodic signal $g(t)$ and a pure tone makes it possible to find the spectral component of $g(t)$ at the frequency of the pure tone.

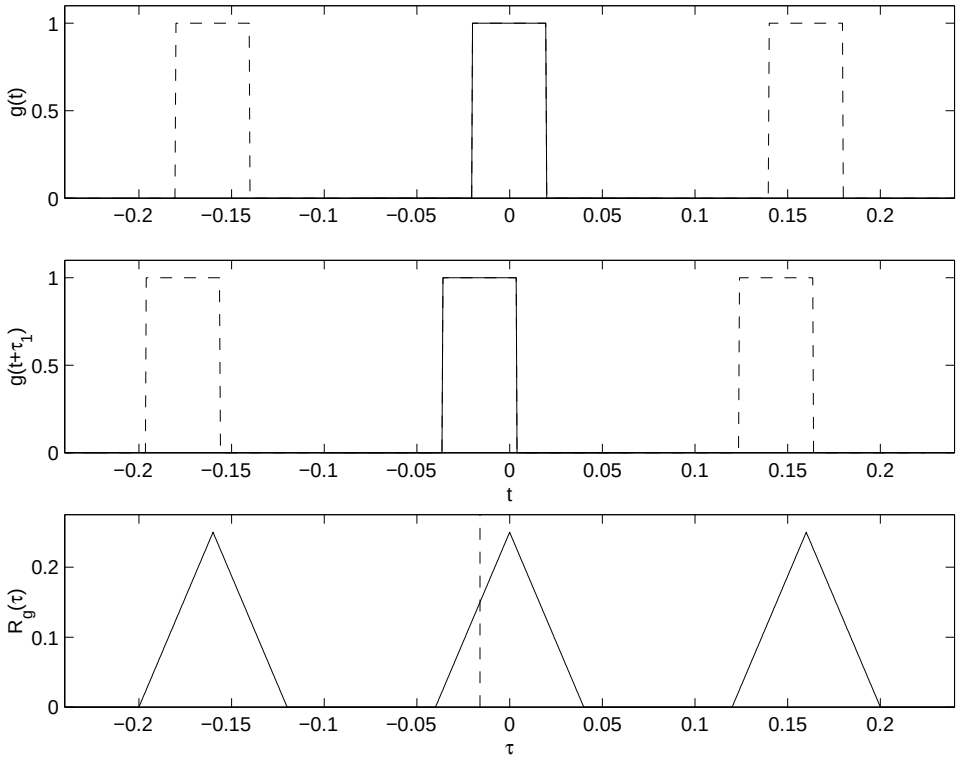


Figure 8.8. Autocorrelation function of a periodic rectangular signal. The figure gives the placement of the signals along the time-axis needed to find $R_g(\tau_1)$. The position of τ_1 is indicated by the dashed line.

Let the spectrum of $g(t)$ be $g(t) \overset{T}{\leftrightarrow} G(m) = G_R(m) + jG_I(m)$, and let the pure tone be

$$\cos 2\pi \frac{m_0}{T} t$$

The cross correlation is

$$R_{gc}(\tau) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} g(t) \cos(2\pi \frac{m_0}{T}(t+\tau)) dt \tag{8.6.33}$$

We can manipulate this expression a bit:

$$\begin{aligned} R_{gc}(\tau) =& \cos 2\pi m_0 \tau/T \cdot \frac{1}{T} \int_{-\frac{1}{2}T}^{\frac{1}{2}T} g(t) \cos 2\pi m_0 t/T dt \\ & - \sin 2\pi m_0 \tau/T \cdot \frac{1}{T} \int_{-\frac{1}{2}T}^{\frac{1}{2}T} g(t) \sin 2\pi m_0 t/T dt \\ =& G_R(m_0) \cos 2\pi m_0 \tau/T + G_I(m_0) \sin 2\pi m_0 \tau/T \end{aligned} \tag{8.6.34}$$

Hence,

$$R_{gc}(0) = G_R(m_0) \qquad \text{and} \qquad R_{gc}\left(\frac{T}{4m_0}\right) = G_I(m_0) \tag{8.6.35}$$

Similar considerations can be carried out for digital signals.

8.7 Linear and non-linear processing of random signals

The most common form of linear processing of random signals in practice, is the filtration with time-invariant filters. The spectral characteristics of random signals are defined with the help of their autocorrelation functions and consequently the description of how filters modify the spectral characteristics of such signals is a bit different than for deterministic signals.

As far as *non-linear signal processing* is concerned, there does not exist a general approach to determining all the characteristics of the result. Non-linear systems with "memory" (i.e. their present non-linear characteristic is dependent on the past of the system and its inputs) pose the largest complications. Such systems will not be treated in this section. Only a couple of very often used in practice systems without memory will be presented.

8.7.1 Filtration of random signals

Many applications involve the filtration of random signals with filters which have an impulse response $h(t)$ or $h(n)$ and a transfer function $H(f)$, where

$$h(t) \longleftrightarrow H(f) \quad \text{and} \quad h(n) \overset{\Delta T}{\longleftrightarrow} H(f) \ . \tag{8.7.1}$$

From a stochastic point of view, this is equivalent with, that for any outcome of the given experiment $x_q(t)$ or $x_q(n)$, a new signal signal $y_q(t)$ or $y_q(n)$ is created by the convolution:

$$y_q(t) = h(t) * x_q(t) \quad \text{or} \quad y_q(n) = h(n) * x_q(n) \ . \tag{8.7.2}$$

In this case $x_q(t)$ or $x_q(n)$ are input signals to the filter. The output of the filter is again a random process $Y(t)$ or $Y(n)$, and it is the properties of this random process, which we will try to find using the properties of the random process $X(t)$ and $X(n)$ and the characteristics of the filter.

It can be shown that if the input signals are strictly stationary, then the output of the filter is stationary too. Conversely, if the input is stationary of kth order, then the output is not necessarily stationary with the same order.

There exists no common calculation procedure to find the probability density function of the output of the filter. It is possible to show, though[5], that if the "duration" of the impulse response of a filter is long compared to "duration" of the autocorrelation function of the input signal, then the probability density function of the resulting signal will approach a normal (Gaussian) distribution. Here the loose term "duration" can be the RMS duration for example.

The mean of $Y(t)$ can be directly derived as:

$$E\{Y(t)\} = \int_{-\infty}^{\infty} h(\Theta) E\{X(t-\Theta)\} d\Theta = E\{X(t)\} \cdot \int_{-\infty}^{\infty} h(\Theta) d\Theta = E\{X(t)\} \cdot H(0) \ , \tag{8.7.3}$$

In other words, the mean of $Y(t)$ is equal to the mean of $X(t)$ multiplied by the DC amplification of the filter. The corresponding expression for the digital signals is:

$$E\{Y(n)\} = \sum_{q=-\infty}^{\infty} h(q) E\{X(n-q)\} = E\{X(n)\} H(0) \ . \tag{8.7.4}$$

The autocorrelation function for the output of the filter for the analog case can be found in the following manner.

From

$$Y(t) = h(t) * X(t) \tag{8.7.5}$$

[5]See e.g. R.A. Altes: Wide Band Systems and Gaussianity, IEEE Transactions on Information Theory, IT-21, November 1975, pages 679-82.

and

$$Y(t+\tau) = h(t) * X(t+\tau) \quad , \tag{8.7.6}$$

one finds that

$$Y(t)Y(t+\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\Theta)X(t-\Theta)\cdot h(\sigma)\cdot X(t+\tau-\sigma)d\Theta d\sigma \quad , \tag{8.7.7}$$

and hereby

$$R_Y(\tau) = E\{Y(t)Y(t+\tau)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\Theta)h(\sigma)R_X(\tau-\sigma+\Theta)d\Theta d\sigma \quad . \tag{8.7.8}$$

By using variable substitution $\gamma = \sigma - \Theta$ we get

$$R_Y(\tau) = \int_{-\infty}^{\infty} R_X(\tau-\gamma)\int_{-\infty}^{\infty} h(\Theta)h(\Theta+\gamma)d\Theta d\gamma \quad . \tag{8.7.9}$$

As mentioned in section 8.5, the inner of the two integrals is the autocorrelation function $R_h(\gamma)$ of the given filter. This means that

$$\boxed{R_Y(\tau) = R_h(\tau) * R_X(\tau)} \quad . \tag{8.7.10}$$

Since

$$R_h(\tau) = h(-\tau) * h(\tau) \leftrightarrow H^*(f)H(f) \quad , \tag{8.7.11}$$

then in frequency domain we get

$$\boxed{S_Y(f) = \mid H(f) \mid^2 S_X(f)} \quad . \tag{8.7.12}$$

These considerations can be carried out also for digital signals and similar results will be obtained.

Cross-correlation function $R_{XY}(\tau)$ between the input and the output signals of the filter can be calculated as

$$\begin{aligned} R_{XY}(\tau) &= E\{X(t)Y(t+\tau)\} = E\{X(t)\int_{-\infty}^{\infty} h(\Theta)X(t+\tau-\Theta)d\Theta\} \\ &= \int_{-\infty}^{\infty} h(\Theta)R_X(\tau-\Theta)d\Theta = h(\tau) * R_X(\tau) \quad . \end{aligned} \tag{8.7.13}$$

In practice, this relation between $R_X(\tau), R_{XY}(\tau)$ and $h(\tau)$ is used to find $h(\tau)$ and hereby $H(f)$ for filters.

In frequency domain, this result is equivalent to

$$S_{XY}(f) = H(f)S_X(f) \quad . \tag{8.7.14}$$

The same expressions are valid for digital signals too.

The *power* P_Y of the output signal can be found in many ways. For analog signals we have:

$$P_Y = \int_{-\infty}^{\infty} R_X(\tau)R_h(\tau)d\tau = \int_{-\infty}^{\infty} \mid H(f) \mid^2 S_X(f)df \quad , \tag{8.7.15}$$

whereas for digital signals we have

$$P_Y = \sum_{q=-\infty}^{\infty} R_X(q)R_h(q) = \frac{1}{f_s}\int_{-f_s/2}^{f_s/2} \mid H(f) \mid^2 S_X(f)df \quad . \tag{8.7.16}$$

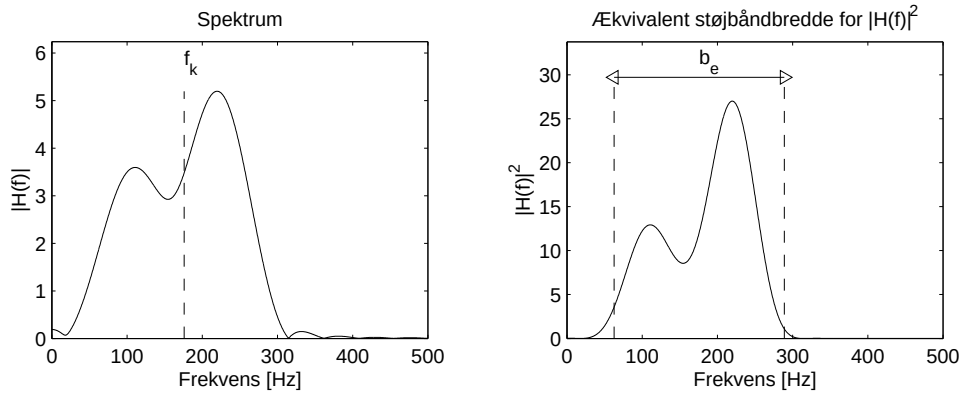


Figure 8.9. Definition of the equivalent noise bandwidth for a filter. Here f_k is chosen as the mean frequency in the spectrum.

Since $P_Y \geqslant 0$ for any filter, this implies that for any random signal the power spectrum can never get negative values, i.e.

$$\underline{S(f) \geqslant 0} \quad . \tag{8.7.17}$$

In connection with filters, an often used measure is the *equivalent noise bandwidth* B_e, which is defined as

$$B_e = \int_0^\infty \frac{\mid H(f) \mid^2}{\mid H(f_k) \mid^2} df \quad \text{and} \quad B_e = \int_0^{f_s/2} \frac{\mid H(f) \mid^2}{\mid H(f_k) \mid^2} df \tag{8.7.18}$$

for analog and digital signals, respectively, where f_k is one or another characteristic frequency of the filter, e.g. $f = 0$, the center frequency or the frequency, where $\mid H(f) \mid$ has a maximum. Although it is not directly evident from the expression, B_e is a function of both $\mid H(f_k) \mid$ and f_k.

The equivalent noise bandwidth for a filter is based on the definition of the equivalent noise bandwidth for a signal $x(t)$. B_e is the width of a fictitious rectangular filter (in frequency domain) filtering noise with uniform power density equal to $S_x(f_k)$ W. B_e is chosen such as the power at the output of the rectangular filter is equal to the power of the signal $x(t)$. The power of $x(t)$ is

$$P_x = \int_0^\infty S_x(f) df \tag{8.7.19}$$

The equivalent power is:

$$P_e = B_e S_x(f_k) \tag{8.7.20}$$

where f_k is the desired frequency. The equivalent bandwidth is easily derived to be:

$$B_e = \frac{1}{S_x(f_k)} \int_0^\infty S_x(f) df \tag{8.7.21}$$

By replacing $S_h(f) = |H(f)|^2$ one gets the expression given in (8.7.18). An example of B_e is shown in Fig. 8.9.

8.7.2 White and colored noise

The term *white noise* refers to noise, which power density spectrum is constant for all frequencies. In other words:

$$S_Y(f) = A \quad , \tag{8.7.22}$$

where A is constant.

The autocorrelation of this noise for analog signals is

$$R_Y(\tau) = A \cdot \delta(\tau) \quad . \tag{8.7.23}$$

The power of such a signal is infinite, implying that such a signal cannot be created physically. This signal has, however, a number of theoretical uses.

The autocorrelation function of digital white noise is

$$R_Y(k) = A\delta(k) \quad , \tag{8.7.24}$$

and the power of this signal is $P_Y = A$. Such a signal can be realized and is often used in digital signal processing.

Band limited white noise is produced by filtering white noise with a filter with an appropriate transfer function $H(f)$. The power density spectrum for this signal is

$$S_X(f) =\mid H(f) \mid^2 A \quad . \tag{8.7.25}$$

The autocorrelation function for this is signal is then

$$R_X(\tau) = AR_h(\tau) \quad . \tag{8.7.26}$$

From the expression for power of analog signals

$$P_X = \int_{-\infty}^{\infty} S_X(f)df = A \int_{-\infty}^{\infty} \mid H(f) \mid^2 d(f) \tag{8.7.27}$$

we get that

$$P_X = A \mid H(f_k) \mid^2 B_e = 2B_e S_X(f_k) \quad , \tag{8.7.28}$$

where B_e is the equivalent noise bandwidth found for the characteristic frequency f_k (see sec. 8.7.1).

From this it appears that B_e can be considered as the bandwidth - for positive frequencies - of the bandlimited white noise signal with the power density $S_X(f_k)$, which has the same power as the original signal.

The derivation for digital signals is similar and the result is

$$P_X = \frac{1}{2f_g} \int_{-f_g}^{f_g} S_X(f)df = \frac{B_e}{f_g} S_X(f_k) \quad , \tag{8.7.29}$$

where f_g is the folding or Nyquist frequency, $f_g = \frac{f_s}{2} = \frac{1}{2\Delta T}$.

The term *pink noise* is used for noise, whose spectrum has the shape

$$S_X(f) = \frac{A}{\mid f \mid} \quad , \tag{8.7.30}$$

where A is a constant. It obvious that the power P_X in this case is also infinitely large. That's why the pink noise signals used in practice are bandlimited both at low and at high frequencies. The spectrum of band-limited pink noise is given by

$$S_X(f) = \begin{cases} \dfrac{A}{\mid f \mid} & \text{for } f_n <\mid f \mid< f_\emptyset \\ 0 & \text{ellers} \end{cases} \tag{8.7.31}$$

and the power of this signal is

$$P_X = 2A \ln \frac{f_\emptyset}{f_n} \quad . \tag{8.7.32}$$

8.7.3 Narrow band noise

The term *narrow band noise* is applied to a signal $V(t)$, whose power density spectrum is not 0 within a narrow frequency band B centered at the frequency f_0. Usually the bandwidth will be much smaller than the center frequency, $B \ll f_0$.

One practical example of narrow band noise is encountered when one tries to determine the spectrum of a random signal with the help of bandpass filters.

As in the case of deterministic narrow-band signals, it is convenient to consider the signal $V(t)$ as a sum of two random signals $X(t)$ and $Y(t)$

$$V(t) = X(t)\cos 2\pi f_0 t + Y(t)\sin 2\pi f_0 t \quad . \tag{8.7.33}$$

Since t is in the argument of the trigonometric factors, it is not obvious that $V(t)$ is stationary, even in the case that $X(t)$ and $Y(t)$ are.

The condition for $V(t)$ to be stationary in wide sense,is that the expected value of $V(t)$ and its autocorrelation function are independent of the zero-point of the time (see sec. 8.2.2). Since

$$E\{V(t)\} = E\{X(t)\}\cos 2\pi f_0 t + E\{Y(t)\}\sin 2\pi f_0 t \quad , \tag{8.7.34}$$

this implies that

$$E\{X(t)\} = 0 \quad \text{and} \quad E\{Y(t)\} = 0 \quad . \tag{8.7.35}$$

From the expression

$$\begin{aligned} R_V(\tau) &= E\{(X(t)\cos 2\pi f_0 t + Y(t)\sin 2\pi f_0 t)(X(t+\tau)\cos 2\pi f_0(t+\tau) \\ &\qquad +Y(t+\tau)\sin 2\pi f_0(t+\tau))\} \\ &= \frac{1}{2}(R_X(\tau)+R_Y(\tau))\cos 2\pi f_0\tau - \frac{1}{2}(R_{YX}(\tau)-R_{YX}(\tau))\sin 2\pi f_0\tau \\ &\quad +\frac{1}{2}(R_X(\tau)-R_Y(\tau))\cos 2\pi f_0(2+\tau) \ +\frac{1}{2}(R_{YX}(\tau)+R_{XY}(\tau))\sin 2\pi f_0(2t+\tau) \quad , \end{aligned} \tag{8.7.36}$$

it follows that if

$$R_X(\tau) = R_Y(\tau) \quad \text{and} \quad R_{XY}(\tau) = -R_{YX}(\tau) \quad , \tag{8.7.37}$$

then $V(t)$ will be wide sense stationary signal and that the following equation is fulfilled:

$$R_V(\tau) = R_X(\tau)\cos 2\pi f_0\tau - R_{YX}(\tau)\sin 2\pi f_0\tau \quad . \tag{8.7.38}$$

From here we can deduce that the spectra of $X(t)$ and $Y(t)$ will be concentrated in a region around $f = 0$ and bandwidth of size on the order of $\frac{1}{2}B$.

If we now create two new random signals

$$A(t) = \sqrt{X^2(t)+Y^2(t)} \qquad \psi(t) = \arctan\left(-\frac{Y(t)}{X(t)}\right) \tag{8.7.39}$$

from the signals $X(t)$ and $Y(t)$, it can be seen that $V(t)$ can be written as

$$V(t) = A(t)\cos(2\pi f_0 t + \psi(t)) \quad . \tag{8.7.40}$$

Also the signals $A(t)$ and $\psi(t)$ will roughly be signals with spectra concentrated around $f = 0$.

If the signals $X(t)$ and $Y(t)$ are mutually *independent and Gaussian*, the one-dimensional probability density functions of both signals $A(t)$ and $\psi(t)$ will be

$$w_A(\alpha) = \begin{cases} \dfrac{\alpha}{\sigma^2} e^{-\frac{\alpha^2}{2\sigma^2}} & \alpha > 0 \\ 0 & \alpha \leqslant 0 \end{cases} \tag{8.7.41}$$

(Rayleigh distribution, see also Table 8.1).

$$\sigma^2 = R_V(0) = P_V \quad , \tag{8.7.42}$$

and[6]

$$w_\psi(\beta) = \begin{cases} \frac{1}{2\pi} & 0 \leqslant \beta \leqslant 2\pi \\ 0 & \text{otherwise} \end{cases} \quad . \tag{8.7.43}$$

$$E\{A(t)\} = (\frac{1}{2}\pi\sigma^2)^{\frac{1}{2}} \tag{8.7.44}$$

and

$$\sigma^2\{A(t)\} = (2 - \frac{1}{2}\pi)\sigma^2 \quad , \tag{8.7.45}$$

and also

$$R_V(\tau) = R_X(\tau)\cos 2\pi f_0\tau \quad . \tag{8.7.46}$$

8.7.4 White noise filtered by a low-pass RC-filter

Let $X(t)$ be analog white noise signal with power density spectrum $S_X(f) = A$, where A is a constant. Let $X(t)$ be filtered by an RC-filter with a transfer function:

$$H_1(f) = \frac{1}{1 + j2\pi fRC} \tag{8.7.47}$$

The power density spectrum of the output signal $Y(t)$ will be

$$S_Y(f) = A \mid H_1(f) \mid^2 = \frac{A}{1 + \left[\frac{f}{f_1}\right]^2} \quad , \tag{8.7.48}$$

where $f_1 = (2\pi RC)^{-1}$.

The autocorrelation function of $S_Y(f)$ is found by applying the inverse Fourier transform on $S_Y(f)$. It is

$$R_Y(\tau) = \frac{A}{2RC}e^{-\frac{|\tau|}{RC}} \tag{8.7.49}$$

(see Fig. 8.10). Then, the total power of the signal is

$$P_Y = \frac{A}{2RC} \quad . \tag{8.7.50}$$

8.7.5 Integrated random signal

In certain applications (e.g. the estimation of the power of a signal) one needs to create the signal

$$Y(t) = \frac{1}{T}\int_{t-T}^{t} X(\theta)d\theta \quad , \tag{8.7.51}$$

from the analog signal $X(t)$.

Since integration over the interval T is equivalent to low-pass filtering, $Y(t)$ can be considered the output of this filter when $X(t)$ is the input signal to the filter. From this immediately follows that

$$E\{Y(t)\} = E\{X(t)\} \quad . \tag{8.7.52}$$

From the shape of the impulse response $h(t)$ of the integrator, shown in Fig. 8.11, it follows that

$$R_h(\tau) = \begin{cases} (1 - \mid \tau \mid /T)/T & \mid \tau \mid < T \\ 0 & \mid \tau \mid > T \end{cases} \tag{8.7.53}$$

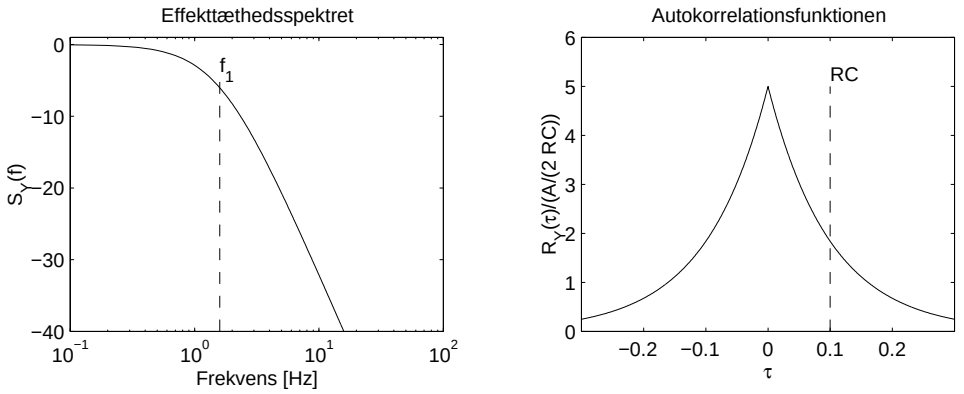


Figure 8.10. The power density spectrum and the autocorrelation function of a RC-filtered white noise.

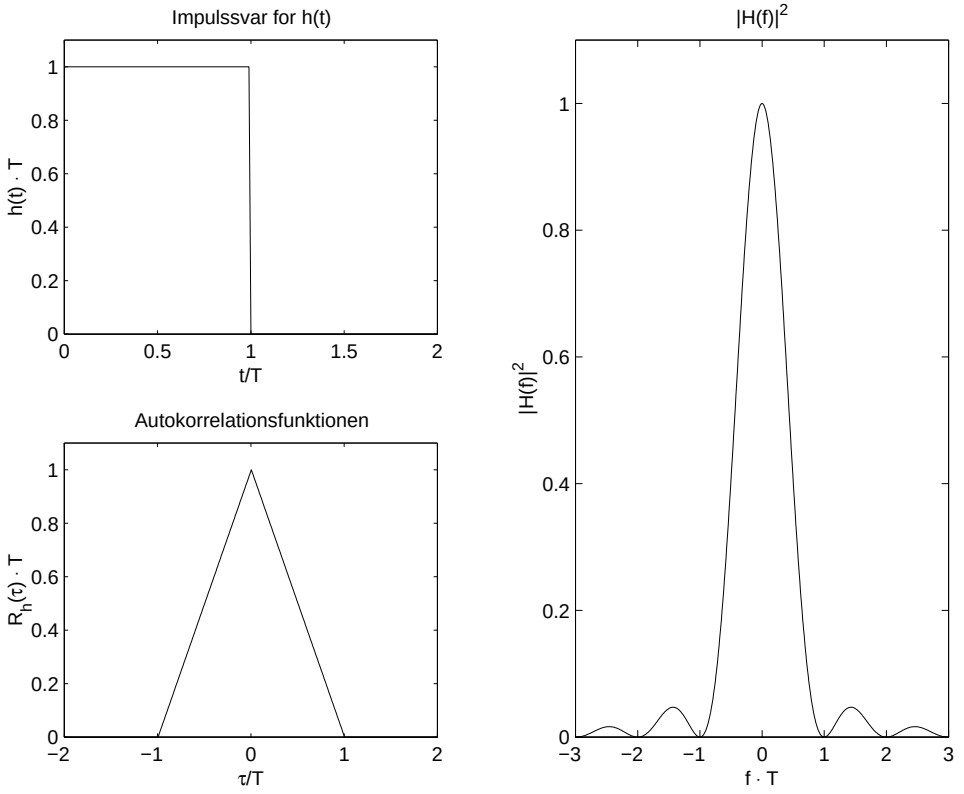


Figure 8.11. Impulse response, autocorrelation function and spectrum of an integrator.

according to which

$$R_Y(\tau) = R_h(\tau) * R_X(\tau) \quad . \tag{8.7.54}$$

If $X(t)$ is white noise signal with a constant power density A, then $R_Y(\tau)$ and $S_Y(f)$ become

$$R_Y(\tau) = AR_h(\tau) \tag{8.7.55}$$

[6]It can be shown, that $A(t)$ and $\psi(t)$ are under these conditions statistically independent

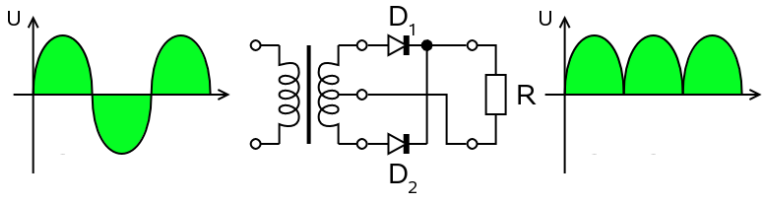


Figure 8.12. A schematic of a full-wave rectifier using a center-tapped transformer.

and

$$S_Y(f) = A\frac{\sin^2 \pi fT}{(\pi fT)^2} \quad , \tag{8.7.56}$$

respectively. The power of the signal $Y(t)$ is

$$P_Y = A/T \quad . \tag{8.7.57}$$

If we consider a digital signal $X(n)$, which is filtered with a digital integrator, then the output signal $Y(n)$ is given by

$$Y(n) = \frac{1}{N} \sum_{q=n-(N-1)}^{n} X(q) \quad . \tag{8.7.58}$$

It follows directly that

$$R_h(k) = \begin{cases} \left(1 - \frac{k}{N}\right)/N & \mid k \mid < N \\ 0 & \text{ellers} \end{cases} \quad . \tag{8.7.59}$$

If $R_X(k) = A\delta(k)$, then $S_X(f) = A$, and with that

$$S_Y(f) = A\left[\frac{1}{N} \cdot \frac{\sin \pi fN\Delta T}{\sin \pi f\Delta T}\right]^2 \tag{8.7.60}$$

The total power of the output signal is

$$P_Y = A/N \quad . \tag{8.7.61}$$

8.7.6 The square-low rectifier

Rectifiers are often used in practice, for example in DC power supplies. These circuits convert a signal which has both negative and positive values to a signal that has only positive values. An example of a *full-wave* rectifier is shown in Fig. 8.12. This section treats a square-low rectifier and the next section a linear one.

The *square-low rectifier* is a non-linear circuit and finds applications in, for example, measuring the power of a random signal. The relation between the input value x and the output value y of the systems is

$$y = ax^2 \quad , \tag{8.7.62}$$

where a is a constant.

Let $X(t)$ be the input signal to the non-linear system, and $Y(t)$ its corresponding output. The mean value of $Y(t)$ is

$$E\{Y(t)\} = aE\{X^2(t)\} = aP_X \quad . \tag{8.7.63}$$

In other words, $Y(t)$ has a DC component, whose magnitude gives the total power of $X(t)$.

If we use the results from section 8.1.3 we find that since

$$x = \pm\sqrt{y/a} \quad , \tag{8.7.64}$$

then

$$w_Y(\eta) = \begin{cases} \frac{w_X(-\sqrt{\eta/a})+w_X(\sqrt{\eta/a})}{2\sqrt{a\eta}} & \eta \geqslant 0 \\ 0 & n < 0 \end{cases} \tag{8.7.65}$$

We will set $a = 1$ in the following for the sake of simplicity.

Given that $X(t)$ *is a Gaussian noise signal* with $E\{X(t)\} = 0$, then

$$w_X(\xi) = \frac{1}{\sqrt{2\pi \cdot R_X(0)}} \cdot e^{-\frac{\xi^2}{2R_X(0)}} \quad , \tag{8.7.66}$$

hence

$$w_Y(\eta) = \begin{cases} \frac{1}{\sqrt{2\pi R_X(0)}} \cdot \frac{1}{\sqrt{\eta}} \cdot e^{-\frac{\eta}{2R_X(0)}} & \eta > 0 \\ 0 & n < 0 \end{cases} \quad . \tag{8.7.67}$$

This probability density function is illustrated in Fig. **??**.

The autocorrelation function $R_Y(\tau)$ of the output signal $Y(t)$ can be found from

$$\begin{aligned} R_Y(\tau) &= E\{Y(t)Y(t+\tau)\} \\ &= E\{X^2(t)X^2(t+\tau)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \xi_1^2\xi_2^2 w_X(\xi_1,\xi_2;\tau)d\xi_1 d\xi_2 \end{aligned} \tag{8.7.68}$$

where $w_X(\xi_1, \xi_2; \tau)$ is as derived in section 8.2.4. If we insert in this expression a new variable:

$$\eta = \xi_2 - \xi_1 \frac{R_X(\tau)}{R_X(0)}$$

then w_X can be expressed as the product of two probability density functions of ξ_1 and η, respectively. This, combined with the fact that $E\{X(t)\} = 0$ and $E\{X^4(t)\} = 3R_X^2(0)$, and some manipulations in time leads to[7]

$$R_Y(\tau) = R_X^2(0) + 2R_X^2(\tau) \quad . \tag{8.7.69}$$

From here it can be seen, that $Y(t)$ has the aforementioned DC-component and the spectrum of $Y(t)$ is

$$S_Y(f) = R_X^2(0)\delta(f) + 2S_X(f) * S_X(f) \quad . \tag{8.7.70}$$

Figure 8.13 shows an example of the spectrum of $Y(t)$, when $X(t)$ is a bandlimited white noise (ideal filter).

It must be noted that the continuous portion of $S_Y(f)$ will always be $\neq 0$ in the vicinity of $f = 0$, independent of the shape of $S_X(f)$.

When the total power of a signal is measured (or its RMS value), the AC-power of the signal $Y(t)$ must also be known. This AC power is

$$P_{AC} = 2R_X^2(0) \quad , \tag{8.7.71}$$

which can be found either from $w_Y(\eta)$, from $R(\tau)$ or from $S_Y(f)$.

[7] This expression appears also from the identity $E\{X_1(t)X_2(t)X_3(t)X_4(t)\} = E\{X_1(t)X_2(t)\}E\{X_3(t)X_4(t)\} + E\{X_1(t)X_3(t)\}E\{X_2(t)X_4(t)\} + E\{X_1(t)X_4(t)\}E\{X_2(t)X_3(t)\}$, which is valid for 4 signals with joint gaussian density function (4-dimensional).

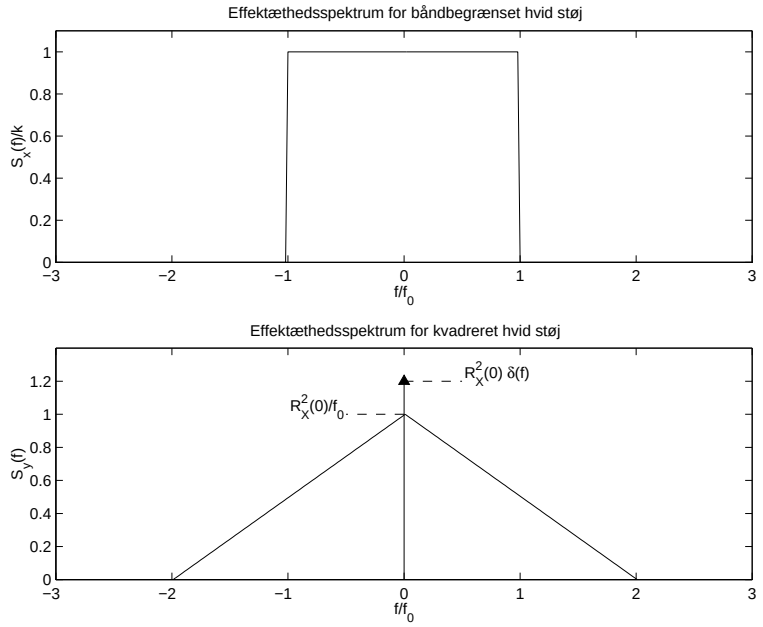


Figure 8.13. Power density spectrum of band-limited white noise, and of squared band-limited white noise.

8.7.7 The linear full-wave (double) rectifier

The linear full-wave rectifier, shown in Fig. 8.12 has a non-linear characteristic given by:

$$y = a \mid x \mid \quad , \tag{8.7.72}$$

where the symbols have the same meaning as in section 8.7.6.

If we also let $X(t)$ be the input signal to and $Y(t)$ the output signal from the rectifier, then the probability density function becomes

$$w_Y(\eta) = \begin{cases} \frac{w_X(-\eta/a)+w_X(\eta/a)}{a} & \eta \geqslant 0 \\ 0 & \eta < 0 \end{cases} \quad . \tag{8.7.73}$$

(see section 8.1.3).

Using $\underline{a = 1}$ and if $\underline{X(t)\ is\ gaussian}$, then the probability density is

$$w_Y(\eta) = \begin{cases} \frac{2}{\sqrt{2\pi R_X(0)}} \cdot e^{-\frac{\eta^2}{2R_X(0)}} & \eta \geqslant 0 \\ 0 & \eta < 0 \end{cases} \quad . \tag{8.7.74}$$

Hence,

$$E\{Y(t)\} = \int_0^\infty \eta \cdot w_Y(\eta) d\eta = \sqrt{\frac{2}{\pi} \cdot R_X(0)} \quad . \tag{8.7.75}$$

This mean value (DC-value) is proportional to $\sqrt{P_X}$.

It can be shown that under these assumptions, the autocorrelation function of $Y(t)$ is

$$R_Y(\tau) = \frac{2}{\pi} R_X(0) + \frac{1}{\pi} \frac{R_X^2(\tau)}{R_X(0)} + \frac{1}{12\pi} \frac{R_X^4(\tau)}{R_X^3(0)} + \cdots \, . \tag{8.7.76}$$

The power density spectrum of $Y(t)$ is found by applying the Fourier transform on the autocorrelation function $R_Y(\tau)$. This spectrum too has a DC-component and an AC-component in the vicinity of the the DC-component (near $f = 0$).

The AC-power of $Y(t)$ is

$$P_{AC} = (1 - \frac{2}{\pi})R_X(0) \quad . \tag{8.7.77}$$