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# FIR Digital Filter Design Techniques Using Weighted Chebyshev Approximation

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*Invited Paper*

***Abstract*—This paper discusses the various approaches to designing FIR digital filters using the theory of weighted Chebyshev approximation. The different design techniques are explained and compared on the basis of their capabilities and limitations. The relationships between filter parameters are briefly discussed for the case of low-pass filters. Extensions of the theory to the problems of magnitude and complex approximation are also included, as are some recent results on the design of two-dimensional FIR filters by transformation.**

## I. Introduction

In the past few years, powerful computer optimization algorithms have been developed to solve the design problem for finite-duration impulse response (FIR) digital filters. It is the purpose of this paper to review these techniques in the light of Chebyshev approximation theory and to describe some of the extensions of this theory.

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FIR digital filters possess certain desirable properties which make them attractive for digital signal processing applications. Among these are the ability to have exactly linear phase and the absence of stability problems in nonrecursive realizations. While long sequences are sometimes necessary to achieve sharp cutoff filters, use of the fast Fourier transform (FFT) can make the realization of such filters computationally competitive even with sharp cutoff infinite-duration impulse response (IIR) elliptic filters.

The process of designing and realizing a digital filter to meet some desired specifications consists of five basic steps.

1) Choose a design technique and convert the desired specifications into a precise mathematical formulation in order to approximate the ideal filter shape.

2) Solve the approximation problem to determine the filter coefficients which minimize a performance measure.

3) Choose a specific structure in which the filter will be realized and quantize the resulting filter coefficients to a fixed word length.

4) Quantize the digital filter variables, i.e., the input, output, and intermediate variable word lengths.

5) Verify by simulation that the resulting design meets given performance specifications.

The results of step 5) generally lead to revisions in steps 1), 2), 3), or 4) in order to meet specifications.

Although it would be desirable to be able to perform steps 2)–4) simultaneously, i.e., to be able to solve the approximation problem for arbitrary structures, with arbitrary word lengths, it is not likely that such a design procedure will be available in the foreseeable future. Thus for the time being we must be content to solve each of these problems independently.

It is quite easy to overlook the importance of step 1) both in the choice of design methods and in the ways that the desired specifications are converted for input to the different design methods. For example, some design algorithms specify the cutoff frequencies and minimize passband and/or stopband deviation whereas other algorithms specify the passband and stopband deviations and obtain the cutoff frequencies only after performing step 2). Algorithms of both types will be described later in this paper.

This paper will be devoted exclusively to a discussion of optimal techniques for solving the approximation problem of step 2). The word "optimal" will denote weighted Chebyshev approximation of a desired frequency response by an FIR digital filter. The presentation will concentrate on linear phase filters where the design techniques are highly developed because the approximation problem is real. We will also discuss the interrelationships between the various design parameters for several filter types. Such relationships are helpful in performing step 1), where the designer must specify parameters such as the filter length, cutoff frequencies, and passband ripple or stopband attenuation. Often these parameters are only implicit in the desired specifications.

Several extensions of the Chebyshev approximation techniques will also be discussed. These include approximation of the magnitude response without regard to the phase, and two-dimensional approximation. Before discussing specific design methods we review various alternative solutions to the FIR filter approximation problem.

## II. History of the FIR Filter Approximation Problem

One of the earliest attempts at deriving the coefficients of an FIR digital filter in order to approximate an ideal desired frequency response was the method of windowing in which the desired frequency response is expanded in a Fourier series and truncated to the desired filter length [1]–[7]. The resulting filter minimizes the least-squares error between the desired response and the filter response. However, the Chebyshev error (the maximum absolute value of the error) from this approach is rather large, due to the Gibbs phenomenon which occurs at discontinuities of the desired frequency response.

Instead of simply truncating the infinite Fourier series, the technique of windowing seeks to reduce the Gibbs phenomenon by multiplying the coefficients of the Fourier series by a smooth time-limited window. Among the more popular windows are the Kaiser window [2], the Hamming window [3], the hanning window [3], and the Dolph–Chebyshev window [4]–[7]. One of the attributes of windowing is that it is an analytical technique, whereas, most other FIR design techniques are iterative in nature.

A second FIR design technique [8]–[14] (the frequency sampling method) was originally proposed by Gold and Jordan [10] and further developed by Rabiner *et al.* [11]. The basic idea behind this method is that one can approximate a specified frequency response by fixing most of its discrete Fourier transform (DFT) coefficients (the frequency samples) and leaving unspecified those DFT coefficients which lie in transition bands. An optimization algorithm is used to choose values for the unspecified coefficients so as to minimize a weighted approximation error over the frequency range of interest. The problem can be shown to be a linear programming problem with very few independent variables, but a large number of constraints.

Herrmann [16] was the first to develop a method for designing optimal (in a Chebyshev sense) FIR filters. By assuming that the frequency response of the optimal low-pass filter was equiripple in both the passband and the stopband, and by fixing the number of ripples in each band, Herrmann was able to write down a set of nonlinear equations which completely described the filter. He then proceeded to solve these equations directly, using an iterative descent method. The length of filters designed in this manner was limited to about 40.

Hofstetter *et al.* [19], [20] removed the restriction on length in Herrmann's approach by developing an algorithm which was "reminiscent of the Remez exchange algorithm" in order to solve the nonlinear equations. In view of later results, it is possible to show that the filters of Herrmann and Hofstetter are a restricted subset of optimal min-max filters, the so-called extraripple or maximal ripple filters. One drawback of the approach of both Herrmann and Hofstetter *et al.* is that it is not possible to specify *a priori* the locations of the passband and stopband cutoff frequencies.

Parks and McClellan [21] formulated the lowpass approximation of the desired response on two disjoint intervals, the passband and the stopband with a transition band left unspecified. Necessary and sufficient conditions for the best Chebyshev approximation were obtained from the classical alternation theorem, and the Remez exchange algorithm was demonstrated to be an effective tool for the computation of these optimal filters. Subsequently, this formulation was extended to include all types of linear phase FIR filters [31].

Rabiner [22], [23] showed that linear programming offered an alternative method for computing the best Chebyshev approximation. Although linear programming is very flexible and can be used to approximate a wide variety of desired filter shapes, it is comparatively slow and hence, the length of the filters it can design is limited.

This paper will describe the last four design methods. Each of these methods yields optimal filters and the theory of Chebyshev approximation provides the underlying mathematical explanation of this behavior. In all cases the filter is restricted to have linear phase so that the approximation problem will be real. In the next section, the linear phase condition for FIR filters is reviewed and the characteristics of the various types of FIR filters are derived.

## III. Characteristics of FIR Filters with Linear Phase

Let $\{h(n)\}$ be a causal finite duration sequence defined over the interval $0 \leqslant n \leqslant N-1$.

The $z$ transform of $\{h(n)\}$ is

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n} = h(0) + h(1)\, z^{-1} + \cdots + h(N-1)\, z^{-(N-1)} \tag{1}$$

TABLE I

| | $\alpha$ | $\beta$ |
|---|---|---|
| Case 1–$N$ odd, Symmetric impulse response | $\frac{N-1}{2}$ | 0 |
| Case 2–$N$ even, Symmetric impulse response | $\frac{N-1}{2}$ | 0 |
| Case 3–$N$ odd, Antisymmetric impulse response | $\frac{N-1}{2}$ | $\frac{\pi}{2}$ |
| Case 4–$N$ even, Antisymmetric impulse response | $\frac{N-1}{2}$ | $\frac{\pi}{2}$ |

and the Fourier transform of $\{h(n)\}$ is

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}. \tag{2}$$

We define a linear phase filter as one whose frequency response can be written in the form

$$H(e^{j\omega}) = G(e^{j\omega})\, e^{j(\beta-\alpha\omega)} \tag{3}$$

where $G(e^{j\omega})$ is real valued. Notice that $G(e^{j\omega})$ is not the magnitude of the frequency response,[1] since $G(e^{j\omega})$ can be negative. Also, (3) only requires the filter to have a constant group delay.

It can be shown that the only solutions for $\alpha$ and $\beta$ are $\alpha = (N-1)/2$ and $\beta = 0$ or $\beta = \pi/2$. When $\beta = 0$ the impulse response is symmetric, i.e., $h(n) = h(N-1-n)$ $n = 0, 1, \cdots, N-1$, and when $\beta = \pi/2$ the impulse response is antisymmetric, i.e., $h(n) = -h(N-1-n)$, $n = 0, 1, \cdots, N-1$. The class of linear phase FIR filters can be divided into four cases according to whether $N$ is odd or even and whether $\beta = 0$ or $\pi/2$. Table I lists these four cases as we will refer to them throughout this paper. It should be noted that when $N$ is even (i.e., cases 2 and 4) the filter group delay $\alpha = (N-1)/2$ is not an integer number of samples. Thus for $N = 10$, $\alpha = 4\frac{1}{2}$ samples delay. This extra "half sample" delay is of importance in some applications, but in most cases it has little effect on the overall processing.

The function $G(e^{j\omega})$ of (3) may be expressed in terms of the impulse response coefficients for each of the four cases of a linear phase filter. Such formulas are derived now for later use in describing various techniques for designing FIR filters to match prescribed frequency response characteristics.

*Case 1:* $N$ odd, symmetric impulse response

$$G(e^{j\omega}) = \sum_{n=0}^{M} a(n) \cos(\omega n) \tag{4}$$

where $M = (N-1)/2$, $a(0) = h(M)$ and $a(n) = 2h(M-n)$ for $n = 1, 2, \cdots, M$.

*Case 2:* $N$ even, symmetric impulse response

$$G(e^{j\omega}) = \sum_{n=1}^{M} b(n) \cos[\omega(n-\tfrac{1}{2})] \tag{5}$$

where $M = N/2$ and $b(n) = 2h(M-n)$ for $n = 1, \cdots, M$.

[1] The terms which are used to refer to frequency are $\omega$ and $f$, with $\omega = 2\pi f$. Throughout this paper the terms $\omega$, and $f$, are used interchangeably.

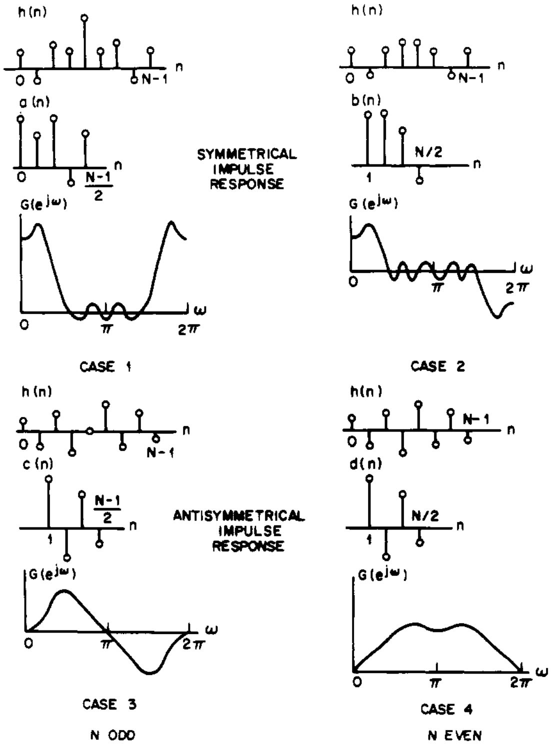


Fig. 1. The four cases of linear phase FIR filters.

*Case 3:* $N$ odd, antisymmetric impulse response

$$G(e^{j\omega}) = \sum_{n=1}^{M} c(n) \sin(\omega n) \tag{6}$$

where $M = (N-1)/2$ and $c(n) = 2h(M-n)$ for $n = 1, 2, \cdots, M$ and $h(M) = 0$.

*Case 4:* $N$ even, antisymmetric impulse response

$$G(e^{j\omega}) = \sum_{n=1}^{M} d(n) \sin[\omega(n-\tfrac{1}{2})] \tag{7}$$

where $M = N/2$ and $d(n) = 2h(M-n)$ for $n = 1, \cdots, M$.

Fig. 1 presents a comprehensive summary of the results of this section. Shown in this figure are typical impulse response sequences $h(n)$, the resulting shifted sequence ($a(n)$ through $d(n)$, depending on the case) and typical frequency response functions $G(e^{j\omega})$, for each of the four cases of linear phase FIR filter.

## IV. Chebyshev Approximation

The formulation of the linear phase FIR design problem as a Chebyshev approximation problem provides a set of conditions (the alternation theorem) which completely characterize the optimal filter. The alternation theorem is the basis of the Remez exchange algorithm which is a very efficient method for calculating the optimal filter coefficients. Furthermore, this characterization has shown that other methods yield a restricted class of optimal filters. In this section we will show how the linear phase FIR design problem can be formulated as a Chebyshev approximation problem and in succeeding sections it will become clear that this formulation is the key to understanding the various FIR design procedures.

The linear phase FIR design problem is an approximation problem in the sense that one is trying to match some ideal amplitude response with the function $G(e^{j\omega})$. For each of

TABLE II

| | $L$ | $G(e^{j\omega})$ |
|---|---|---|
| Case 1–$N$ odd, Symmetric impulse response | 0 | $\sum_{n=0}^{(N-1)/2} a(n)\cos(\omega n)$ |
| Case 2–$N$ even, Symmetric impulse response | 0 | $\sum_{n=1}^{N/2} b(n)\cos[\omega(n-1/2)]$ |
| Case 3–$N$ odd, Antisymmetric impulse response | 1 | $\sum_{n=1}^{(N-1)/2} c(n)\sin(\omega n)$ |
| Case 4–$N$ even, Antisymmetric impulse response | 1 | $\sum_{n=1}^{N/2} d(n)\sin[\omega(n-1/2)]$ |

TABLE III

| | $Q(e^{j\omega})$ | $P(e^{j\omega})$ |
|---|---|---|
| Case 1 | 1 | $\sum_{n=0}^{M} a(n)\cos(\omega n)$ |
| Case 2 | $\cos(\omega/2)$ | $\sum_{n=0}^{M-1} \tilde{b}(n)\cos(\omega n)$ |
| Case 3 | $\sin(\omega)$ | $\sum_{n=0}^{M-1} \tilde{c}(n)\cos(\omega n)$ |
| Case 4 | $\sin(\omega/2)$ | $\sum_{n=0}^{M-1} \tilde{d}(n)\cos(\omega n)$ |

the four cases the function $G(e^{j\omega})$ is a linear comoination of a different set of basis functions. It is convenient to reduce all cases to a common one with cosine basis functions.

In the previous section on linear phase FIR filters, it was shown that the frequency response of the four cases of linear phase filters could be written in the form

$$H(e^{j\omega}) = e^{-j\omega[(N-1)/2]}\, e^{j(\pi/2)L}\, G(e^{j\omega}). \tag{8}$$

Values for $L$ and the form for $G(e^{j\omega})$ are given in Table II for each of the four cases of linear phase.

Using simple trigonometric identities, each of the expressions for $G(e^{j\omega})$ in Table II can be written as a product of a fixed function of $\omega$ (call this $Q(e^{j\omega})$), and a term which is a sum of cosines (call this $P(e^{j\omega})$). Thus the expressions for $G(e^{j\omega})$ in Table II become [31] as follows.

*Case 1:*
No change.

*Case 2:*

$$\sum_{n=1}^{M} b(n)\cos[\omega(n-1/2)] = \cos(\omega/2)\sum_{n=0}^{M-1}\tilde{b}(n)\cos(\omega n). \tag{9}$$

*Case 3:*

$$\sum_{n=1}^{M} c(n)\sin(\omega n) = \sin(\omega)\sum_{n=0}^{M-1}\tilde{c}(n)\cos(\omega n). \tag{10}$$

*Case 4:*

$$\sum_{n=1}^{M} d(n)\sin[\omega(n-1/2)] = \sin(\omega/2)\sum_{n=0}^{M-1}\tilde{d}(n)\cos(\omega n). \tag{11}$$

The coefficients $\tilde{b}(n)$, $\tilde{c}(n)$, and $\tilde{d}(n)$ in (9)–(11) are simply related to $b(n)$, $c(n)$, and $d(n)$ of Table II as shown in [31]. Table III shows a summary of the functions $Q(e^{j\omega})$ and $P(e^{j\omega})$ for each of the four cases of linear phase filters.

For Cases 2-4, $Q(e^{j\omega})$ is constrained to be zero at either $\omega = 0$ or $\omega = \pi$, or both.

To formulate the optimal linear phase FIR filter design problem as a Chebyshev approximation problem, it is necessary to define $D(e^{j\omega})$, the desired (real) frequency response of the filter, and $W(e^{j\omega})$, a weighting function on the approximation error which enables the designer to choose the relative size of the error in different frequency bands. The weighted error of approximation $E(e^{j\omega})$ is, by definition,

$$E(e^{j\omega}) = W(e^{j\omega})\,[D(e^{j\omega}) - G(e^{j\omega})]. \tag{12}$$

By writing $G(e^{j\omega})$ as a product of $P(e^{j\omega})$ and $Q(e^{j\omega})$, $E(e^{j\omega})$ can be rewritten as

$$E(e^{j\omega}) = W(e^{j\omega})\,[D(e^{j\omega}) - P(e^{j\omega})\,Q(e^{j\omega})]. \tag{13}$$

Since $Q(e^{j\omega})$ is a fixed function of frequency, it can be factored out of (13), giving

$$E(e^{j\omega}) = W(e^{j\omega})\,Q(e^{j\omega})\left[\frac{D(e^{j\omega})}{Q(e^{j\omega})} - P(e^{j\omega})\right]. \tag{14}$$

Equation (14) is valid everywhere except possibly at $\omega = 0$ and/or $\omega = \pi$. Defining $\hat{W}(e^{j\omega})$ and $\hat{D}(e^{j\omega})$ as

$$\hat{W}(e^{j\omega}) = W(e^{j\omega})\,Q(e^{j\omega}) \tag{15}$$

and

$$\hat{D}(e^{j\omega}) = \frac{D(e^{j\omega})}{Q(e^{j\omega})} \tag{16}$$

the error function may be written as

$$E(e^{j\omega}) = \hat{W}(e^{j\omega})\,[\hat{D}(e^{j\omega}) - P(e^{j\omega})]. \tag{17}$$

The Chebyshev approximation problem may now be stated as finding the set of coefficients ($a(n)$, $\tilde{b}(n)$, $\tilde{c}(n)$, or $\tilde{d}(n)$) to minimize the maximum absolute value of $E(e^{j\omega})$ over the frequency bands in which the approximation is being performed. Using the notation $\|E(e^{j\omega})\|$ to denote this minimum value (i.e., the $L_\infty$–norm of $E(e^{j\omega})$), the Chebyshev approximation problem may be stated mathematically as

$$\|E(e^{j\omega})\| = \min_{\{\text{coefficients}\}}\,[\max_{\omega\in A} |E(e^{j\omega})|] \tag{18}$$

where $A$ represents the disjoint union of all the frequency bands of interest.

A well-known property of this class of Chebyshev approximation problems may be used to obtain a characterization of the solution to (18). This is the so-called alternation theorem which may be stated as follows [41].

*Theorem:* If $P(e^{j\omega})$ is a linear combination of $r$ cosine functions (i.e., $P(e^{j\omega}) = \sum_{n=0}^{r-1} \alpha(n)\cos(\omega n)$) then a necessary and sufficient condition that $P(e^{j\omega})$ be the unique, best weighted Chebyshev approximation to a continuous function $\hat{D}(e^{j\omega})$ on $A$, a compact subset of $[0, \pi]$, is that the weighted error function $E(e^{j\omega})$ exhibit at least $r + 1$ extremal frequencies in $A$, i.e., there must exist $r + 1$ points $\omega_i$ in $A$ such that $\omega_1 < \omega_2 < \cdots < \omega_r < \omega_{r+1}$ and such that $E(e^{j\omega_i}) = -E(e^{j\omega_{i+1}})$, $i = 1, 2, \cdots, r$, and $|E(e^{j\omega_i})| = \max_{\omega \in A} |E(e^{j\omega})|$.

The preceding alternation theorem is extremely powerful in that it expresses a necessary and sufficient set of conditions for obtaining the optimal Chebyshev solution. A number of techniques have been devised for obtaining this optimal solution, depending on the interpretation of this theorem.

It is worth noting that the alternation theorem depends very strongly on the fact that the basis functions satisfy the Haar condition [41]. When one attempts to do constrained Chebyshev approximation or two-dimensional approximation, it turns out that the basis functions do not satisfy the Haar condition. Thus, a characterization of the optimal solution in the form of an alternation theorem is no longer possible. Before discussing any specific algorithm for designing optimal filters, the next section presents an important result on the maximum number of extrema of a linear phase FIR filter.

### *Constraint on the Number of Extrema of the Frequency Response of a Linear Phase Filter*

The alternation theorem states that for the optimal linear phase FIR filter, the error function has at least $r + 1$ extrema where $r$ is the number of cosine functions being used in the approximation. Since for many cases of interest the extrema of $G(e^{j\omega})$ are also the extrema of $E(e^{j\omega})$, (i.e., both $dW(e^{j\omega})/d\omega$ and $dD(e^{j\omega})/d\omega$ will be zero when $dG(e^{j\omega})/d\omega$ is zero), it is important to know the maximum number of extrema of $G(e^{j\omega})$. By adding to this number the number of extrema of $E(e^{j\omega})$ which are not extrema of $G(e^{j\omega})$ the total maximum number of extrema of $E(e^{j\omega})$ can be found.

By differentiating $G(e^{j\omega})$ with respect to $\omega$, it can be shown that $N_e$, the number of extrema of $G(e^{j\omega})$, obeys

$$\begin{aligned} N_e &\leq (N+1)/2 && \text{case 1}\\ N_e &\leq N/2 && \text{case 2}\\ N_e &\leq (N-1)/2 && \text{case 3}\\ N_e &\leq N/2 && \text{case 4.} \end{aligned} \tag{19}$$

Equation (19) only constrains the number of extrema of $G(e^{j\omega})$. It is readily seen that if the approximation problem is being solved over a union of disjoint frequency bands, the error function can obtain an extremum at *each* band edge, whereas these points will generally *not* be extrema of $G(e^{j\omega})$ [21]. The exception to this rule is when the band edges are at either $\omega = 0$ or $\omega = \pi$ where $G(e^{j\omega})$ will often have an extremum. Thus, for example, the error function for Case 1 lowpass filter (a two-band approximation problem) can have a maximum of $(N + 5)/2$ extrema, i.e., $(N + 1)/2$ extrema of $G(e^{j\omega})$ and 2 extra extrema for the passband and stopband edges. The error function for a Case 1 bandpass filter (a three-band approximation problem) can have a maximum of $(N + 9)/2$ extrema, i.e., $(N + 1)/2$ extrema of $G(e^{j\omega})$ and 4 extra extrema for the passband and stopband edges.

Foreknowledge of the maximum number of extrema of $E(e^{j\omega})$ is important because it relates to the exact ways in which design techniques have been devised to design optimal filters. For example, two of the optimal design techniques are only capable of designing optimal filters with the *maximum* possible number of extrema. These design techniques are of limited utility in that the alternation theorem shows that, in general, filters with the maximum number of extrema in their error functions are special cases of the theorem, and hence are only a subset of the larger class of optimal filters. In the following sections a discussion of the various optimal filter design algorithms is given. Both for historical reasons, and for development purposes, we describe first the two algorithms which only are capable of designing optimal filters with the maximum possible number of extrema in their error functions. Then a discussion is given of a Remez type algorithm and finally a linear programming method for designing any optimal, linear phase, FIR filter.

## V. Maximal Ripple FIR Filters

In the preceding section, it was shown that the number of frequencies at which $G(e^{j\omega})$ could attain an extremum is strictly a function of the case of linear phase filter under investigation. At each extremum, the value of $G(e^{j\omega})$ is predetermined by a combination of the weighting function $W(e^{j\omega})$, the desired frequency response $D(e^{j\omega})$, and the quantity $\delta$ which represents the peak error of approximation. By distributing the frequencies at which $G(e^{j\omega})$ attained an extremal value among the different frequency bands over which a desired response was being approximated, and by requiring the resulting filter to have the maximum number of extremal frequencies, a unique optimal filter can be obtained. Since these filters have the maximum number of alternations, or ripples, in their error of approximation curve, they have been called maximal ripple filters. For the case of lowpass filters these maximal ripple filters have also been called extraripple filters [21] because only a single extra ripple above the minimum number required for optimality is present.

The manner in which a set of nonlinear equations is obtained for describing the maximal ripple filter is as follows. At each of the $N_e$ unknown extremal frequencies, $E(e^{j\omega})$ attains the maximum value of $\pm\delta$, and $E(e^{j\omega})$, or equivalently, $G(e^{j\omega})$ has zero derivative. Thus $2N_e$ equations of the form

$$G(e^{j\omega_i}) = \frac{\pm\delta}{W(e^{j\omega_i})} + D(e^{j\omega_i}), \qquad i = 1, 2, \cdots, N_e$$

$$\frac{d}{d\omega}[G(e^{j\omega})]\bigg|_{\omega=\omega_i} = 0, \qquad i = 1, 2, \cdots, N_e$$

for $i = 1, 2, \cdots, N_e$ are obtained. These equations represent a set of $2N_e$ nonlinear equations in $2N_e$ unknowns ($N_e$ impulse response coefficients, and $N_e$ frequencies at which $G(e^{j\omega})$ obtains the extremal value). The set of $2N_e$ equations may be solved iteratively using a nonlinear optimization procedure such as the well-known Fletcher–Powell algorithm.

Two facts should be noted about this procedure. First the quantity $\delta$ (i.e., the peak error) is a fixed quantity and is not minimized by the optimization scheme. Thus the shape of $G(e^{j\omega})$ is postulated *a priori*, and only the frequencies at which $G(e^{j\omega})$ attains the extremal values are unknown. Second, the design procedure has no way of specifying band edges for the different frequency bands of the filter. Thus the optimization algorithm does not work on given frequency bands, but instead is free to select exactly where the bands will lie. This lack of control over frequency band edges diminishes the utility of this and the next algorithm to be discussed.

To illustrate a specific set of equations for optimization we consider the design of a case 1 low-pass filter with $N = 15$, a

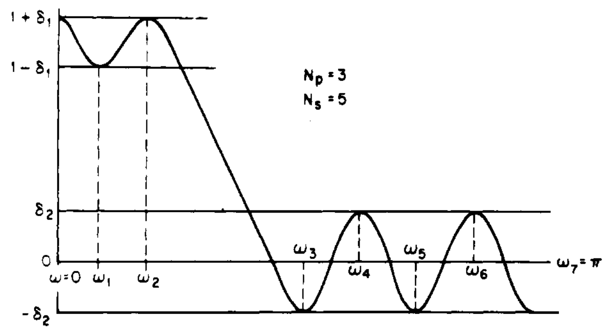


Fig. 2. The frequency response of a maximal ripple low-pass filter.

peak ripple of $\delta = \delta_2$, a weighting function defined as

$$W(e^{j\omega}) = \begin{cases} \delta_2/\delta_1, & \omega \text{ in the passband} \\ 1, & \omega \text{ in the stopband} \end{cases}$$

and a desired response of

$$D(e^{j\omega}) = \begin{cases} 1, & \omega \text{ in the passband} \\ 0, & \omega \text{ in the stopband.} \end{cases}$$

Fig. 2 illustrates $G(e^{j\omega})$ for this example. The extremal frequencies are the set $\omega = 0$, $\omega_1$, $\omega_2$, $\omega_3$, $\omega_4$, $\omega_5$, $\omega_6$, and $\omega_7 = \pi$. At $\omega = 0$ and $\omega = \pi$, $G(e^{j\omega})$ has zero derivative, independent of the impulse response coefficients for a case 1 design. For this example, the $N_e = 8$ extremal frequencies are divided so that $N_p = 3$ occur in the passband, and $N_s = 5$ occur in the stopband. Thus for this set of conditions, the following equations are obtained:

| Function Constraints | Derivative Constraints |
|---|---|
| $G(e^{j0}) = 1 + \delta_1$ | |
| $G(e^{j\omega_1}) = 1 - \delta_1$ | $\left.\frac{d}{d\omega} G(e^{j\omega})\right\vert_{\omega=\omega_1} = 0$ |
| $G(e^{j\omega_2}) = 1 + \delta_1$ | $\left.\frac{d}{d\omega} G(e^{j\omega})\right\vert_{\omega=\omega_2} = 0$ |
| $G(e^{j\omega_3}) = -\delta_2$ | $\left.\frac{d}{d\omega} G(e^{j\omega})\right\vert_{\omega=\omega_3} = 0$ |
| $G(e^{j\omega_4}) = +\delta_2$ | $\left.\frac{d}{d\omega} G(e^{j\omega})\right\vert_{\omega=\omega_4} = 0$ |
| $G(e^{j\omega_5}) = -\delta_2$ | $\left.\frac{d}{d\omega} G(e^{j\omega})\right\vert_{\omega=\omega_5} = 0$ |
| $G(e^{j\omega_6}) = +\delta_2$ | $\left.\frac{d}{d\omega} G(e^{j\omega})\right\vert_{\omega=\omega_6} = 0.$ |
| $G(e^{j\pi}) = -\delta_2.$ | |

Once this set of equations has been solved for the unknown frequencies and the impulse-response coefficients, the passband and stopband edges may be solved for by searching for the frequency beyond $\omega_2$ where $G(e^{j\omega})$ exactly equals $1 - \delta_1$ (passband edge), and the frequency before $\omega_3$ where $G(e^{j\omega})$ exactly equals $+\delta_2$ (stopband edge).

The preceding optimization procedure has been used by Herrmann [16] to design low-pass and bandpass filters for values of $N$ up to about 61. The next section discusses another technique for designing maximal ripple filters where much larger filters can be designed efficiently.

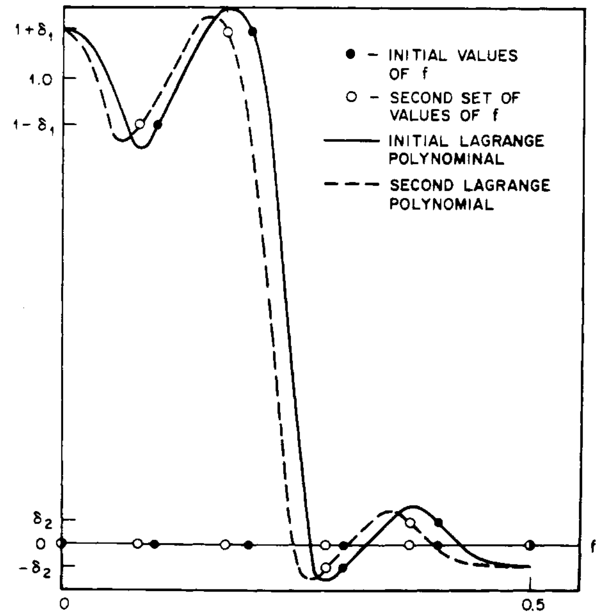


Fig. 3. Polynomial interpolation iterative solution for maximal ripple low-pass filter.

### *Polynomial Interpolation Solution for Maximal Ripple FIR Filters*

A second, more efficient, method has been proposed for designing maximal ripple filters. This algorithm is basically an iterative technique for producing a polynomial ($G(e^{j\omega})$) that has extrema of desired values. The algorithm begins by making an initial estimate of the frequencies at which the extrema in $G(e^{j\omega})$ will occur, and then uses the well-known Lagrange interpolation formula to obtain a polynomial that alternatingly goes through the maximum allowable ripple values at these frequencies. It has been experimentally found that the initial guess of extremal frequencies does not affect the ultimate convergence of the algorithm, but instead affects the number of iterations required to achieve the desired result.

Rather than consider the general filter design problem, it is instructive to consider the design of a case 1 low-pass filter as an example of how the algorithm works. Fig. 3 shows the frequency response of a low-pass filter with $N = 11$, peak ripple $\delta = \delta_2$, weighting function $W(e^{j\omega})$, and desired frequency response $D(e^{j\omega})$, as defined in the preceding section. The number of extremal frequencies $N_e$ is 6 for this example, and they are divided into $N_p = 3$ passband extrema, and $N_s = 3$ stopband extrema. The filled dots along the frequency axis of Fig. 3 show the initial guess as to the extremal frequencies of $G(e^{j\omega})$. The solid line shows the initial Lagrange polynomial obtained by choosing polynomial coefficients so that the values of the polynomial at the guessed set of frequencies are identical to the assigned extreme values. As seen in Fig. 3, the polynomial associated with the initial guess does not have extrema that achieve the maximum allowable errors, but rather it has extrema that exceed these values. The next stage of the algorithm is to locate the frequencies at which the extrema of the first Lagrange interpolation occur. These frequencies are used as an updated improved guess of the frequencies for which the extrema of the filter response will achieve the desired ripple values. This second set of frequencies is indicated by the open dots in Fig. 3. The algorithm uses these new frequencies to construct another Lagrange polynomial (shown by the dotted line in Fig. 3) that achieves the desired values at these frequencies. At this point the iterative nature of the

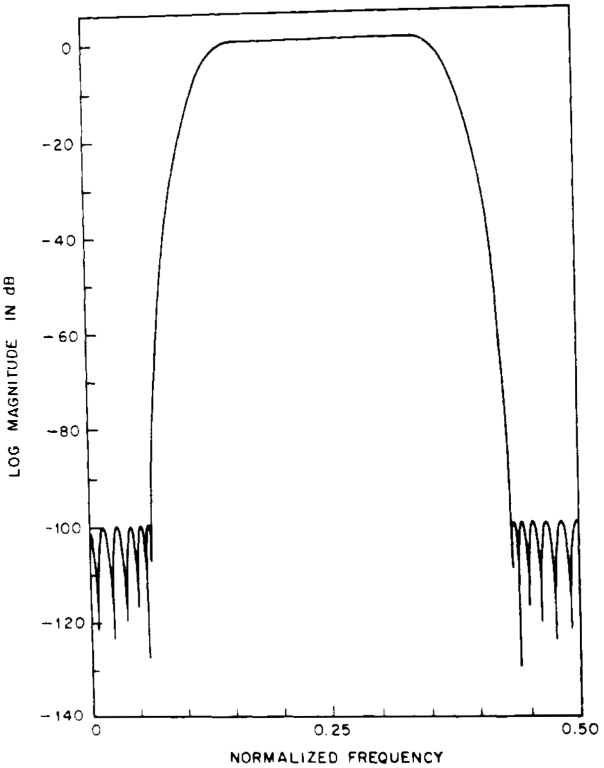


Fig. 4. The frequency response of a maximal ripple bandpass filter.

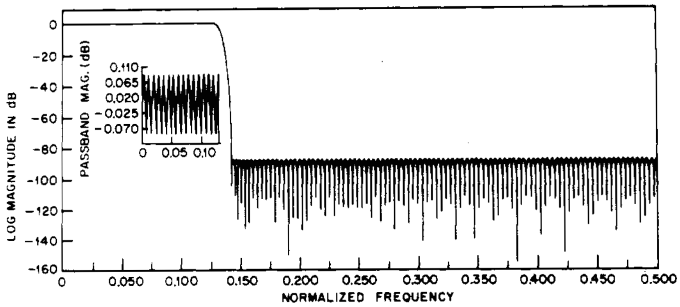


Fig. 5. The frequency response of a maximal ripple low-pass filter.

algorithm has emerged. By locating the extrema of the new polynomial, another iteration of the algorithm is begun. This algorithm is quite similar to the well-known Remez multiple exchange algorithm of Chebyshev approximation theory.

Two typical maximal ripple filters designed by Hofstetter *et al.* [19], [20] using this algorithm are shown in Figs. 4 and 5. Figure 4 shows the log magnitude response of a case 1 bandpass filter with $N = 41$, (i.e., $N_e = 21$) with 6 extrema of $G(e^{j\omega})$ in each stopband, and 9 extrema in the passband. The peak ripple in the stopbands is $\delta_2 = 0.00001$ (or $-100$ dB), whereas the peak ripple in the passband is 0.005. Fig. 5 shows the log magnitude response of a case 1 low-pass filter with $N = 251$ and with 33 extrema of $G(e^{j\omega})$ in the passband and 94 extrema in the stopband. The peak ripple in the passband is $\delta_1 = 0.01$ and the peak ripple in the stopband is $\delta_2 = 0.00004$ (or $-88$ dB).

Although this improved algorithm has essentially eliminated the difficulty of designing filters with large values of $N$, the inherent problem still remains that the filter band edge frequencies cannot be specified *a priori*, i.e., they must be calculated from the final solution. Furthermore, both techniques are only capable of designing maximal ripple filters, which, as discussed earlier, are a subclass of the class of optimal filters. In the next sections design techniques are presented which are capable of designing any optimal filter.

## VI. Remez Exchange Algorithm Design of Optimal FIR Filters

As shown earlier, the optimal linear phase FIR filter design problem is a Chebyshev approximation problem where the approximating function $P(e^{j\omega})$ in (17) is a sum of $r$ independent cosine functions. The alternation theorem gave a set of necessary and sufficient conditions on the weighted error function $E(e^{j\omega})$ (see (17)) such that the solution was the unique best approximation to the desired frequency response $\hat{D}(e^{j\omega})$. The Remez exchange algorithm is an algorithm which solves the Chebyshev approximation problem by searching for the extremal frequencies of the best approximation. This is accomplished as follows. At the beginning of each iteration one has a set of $r + 1$ extremal frequencies $\{\omega_k\}$. Equation (20) gives the set of equations which must be solved for the generalized polynomial approximating function $P(e^{j\omega})$ whose weighted error function has magnitude $\delta$ with alternating signs on the set $\{\omega_k\}$

$$\hat{W}(e^{j\omega_k})[\hat{D}(e^{j\omega_k}) - P(e^{j\omega_k})] = (-1)^k \delta, \quad k = 0, 1, \cdots, r \tag{20}$$

where $P(e^{j\omega})$ is of the form

$$P(e^{j\omega}) = \sum_{n=0}^{r-1} \alpha(n) \cos(\omega n).$$

Equation (20) can be rewritten in matrix form as shown in (21). The invertibility of this matrix is guaranteed by the Haar condition on the basis functions.

$$\begin{bmatrix} 1 & \cos\omega_0 & \cos 2\omega_0 & \cdots & \cos[(r-1)\omega_0] & \dfrac{1}{\hat{W}(e^{j\omega_0})} \\ 1 & \cos\omega_1 & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 1 & \cos\omega_r & & & & \dfrac{(-1)^r}{\hat{W}(e^{j\omega_r})} \end{bmatrix} \begin{bmatrix} \alpha(0) \\ \alpha(1) \\ \vdots \\ \alpha(r-1) \\ \delta \end{bmatrix} = \begin{bmatrix} \hat{D}(e^{j\omega_0}) \\ \hat{D}(e^{j\omega_1}) \\ \vdots \\ \\ \hat{D}(e^{j\omega_r}) \end{bmatrix}. \tag{21}$$

Since direct solution of (21) is both difficult, and slow, it is more efficient to calculate $\delta$ analytically as

$$\delta = \frac{a_0 \hat{D}(e^{j\omega_0}) + a_1 \hat{D}(e^{j\omega_1}) + \cdots + a_r \hat{D}(e^{j\omega_r})}{a_0/\hat{W}(e^{j\omega_0}) - a_1/\hat{W}(e^{j\omega_1}) + \cdots + (-1)^r a_r/\hat{W}(e^{j\omega_r})} \tag{22}$$

where

$$a_k = \prod_{\substack{i=0 \\ i \neq k}}^{r} \frac{1}{(x_k - x_i)} \tag{23}$$

$$x_i = \cos \omega_i. \tag{24}$$

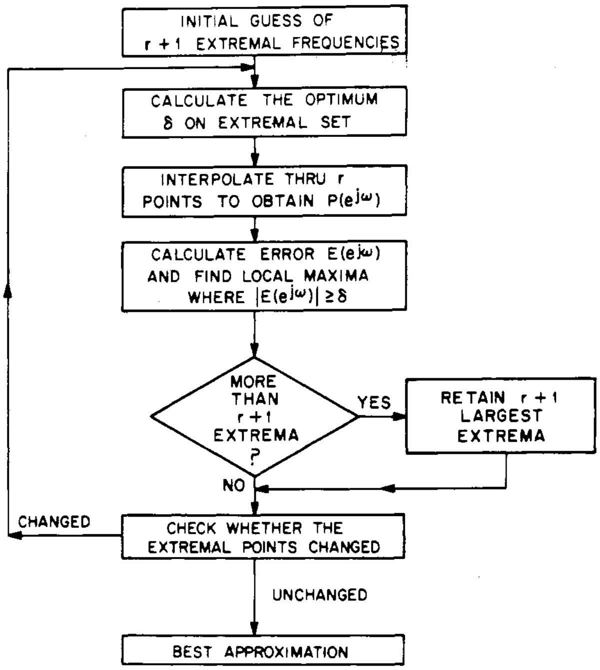


**Fig. 6. Flow chart of the Remez exchange algorithm.**

After calculating $\delta$, the Lagrange interpolation formula in the barycentric form[2] is used to interpolate $P(e^{j\omega})$ on the $r$ points $\omega_0, \omega_1, \cdots, \omega_{r-1}$ to the values

$$C_k = \hat{D}(e^{j\omega_k}) - (-1)^k \frac{\delta}{\hat{W}(e^{j\omega_k})}, \qquad k = 0, 1, \cdots, r-1 \quad (25)$$

$$P(e^{j\omega}) = \frac{\sum_{k=0}^{r-1} \left[\frac{\beta_k}{x - x_k}\right] C_k}{\sum_{k=0}^{r-1} \left[\frac{\beta_k}{x - x_k}\right]} \quad (26)$$

where

$$\beta_k = \prod_{\substack{i=0 \\ i \neq k}}^{r-1} \frac{1}{(x_k - x_i)}. \quad (27)$$

Note that $P(e^{j\omega})$ will also interpolate to $D(e^{j\omega_r}) - [(-1)^r \delta / \hat{W}(e^{j\omega_r})]$ since it satisfies (20). The next step in the process is to evaluate $E(e^{j\omega})$ on a dense set of the frequency axis. If $|E(e^{j\omega})| \leq \delta$ for all frequencies in the dense set, then the optimal approximation has been found. If $|E(e^{j\omega})| > \delta$ for some frequencies in the dense set, then a new set of $r + 1$ frequencies must be chosen as candidates for the extremal frequencies. The new points are chosen as the peaks of the resulting error curve, thereby forcing $\delta$ to increase and ultimately converge to its upper bound which corresponds to the solution to the problem. In the event that there are more than $r + 1$ extrema in $E(e^{j\omega})$ at any iteration, the $r + 1$ frequencies at which $|E(e^{j\omega})|$ is largest are retained as the guessed set of extremal frequencies for the next iteration. Fig. 6 summarizes the exchange algorithm in a block diagram form.

[2]See R. W. Hamming, *Numerical Methods for Scientists and Engineers*, 1st ed. New York: McGraw-Hill, for a discussion of the barycentric form.

The filter impulse response is obtained by evaluating $P(e^{j\omega})$ at $N$ equally spaced frequencies and using the DFT to get the sequence $\{\alpha(n)\}$, from which the impulse response coefficients may be derived. Depending on which case linear phase filter is derived, a unique formula can be written for obtaining $h(n)$ from $\alpha(n)$.

A general purpose computer program has been written to implement this algorithm and has found widespread use in filter design applications [34].

## VII. Linear Programming Design of Optimal FIR Filters

The optimal linear phase FIR filter is the one for which the maximum error $E(e^{j\omega})$ is minimized over all $\omega$. Letting $\delta$ represent the maximum error, a set of linear inequalities can be written to describe this minimax problem, i.e.,

$$-\delta \leq \hat{W}(e^{j\omega_i}) [\hat{D}(e^{j\omega_i}) - P(e^{j\omega_i})] \leq \delta, \qquad \omega_i \in F \quad (28)$$

where $F$ is a dense grid of frequencies in the bands over which the approximation is being made. Equation (28) can formally be written as a linear program, i.e.,

$$-\hat{W}(e^{j\omega_i}) \sum_{m=0}^{r-1} \alpha(m) \cos(m\omega_i) - \delta \leq -\hat{W}(e^{j\omega_i}) \hat{D}(e^{j\omega_i})$$

$$\hat{W}(e^{j\omega_i}) \sum_{m=0}^{r-1} \alpha(m) \cos(m\omega_i) - \delta \leq \hat{W}(e^{j\omega_i}) \hat{D}(e^{j\omega_i})$$

minimize $\delta$.

Linear programming techniques can be used to solve the preceding set of equations [22], [23]. However, since linear programming is basically a single exchange method, it is significantly slower than the Remez method, and hence is avoided for this class of problems. In a later section, however, it will be shown how when time response constraints are added to the design problem, linear programming is perhaps the only simple method of solving the problem.

## VIII. Characteristics of Optimal Case 1 Low-Pass Filters

For a low-pass filter the optimal design problem consists of specifying the filter length $N$, the passband cutoff frequency $F_p$, the stopband cutoff frequency $F_s$, and the ripple ratio $K = \delta_1/\delta_2$ which describes the desired weighting function $W(e^{j\omega})$ as

$$W(e^{j\omega}) = \begin{cases} 1/K = \delta_2/\delta_1, & 0 \leq \omega \leq 2\pi F_p \\ 1, & 2\pi F_s \leq \omega \leq \pi \end{cases} \quad (29)$$

where $\delta_1$ is the passband ripple, and $\delta_2$ is the stopband ripple. Fig. 7 shows the frequency response of a Case 1 low-pass filter. The auxiliary parameter $\Delta F$ is defined as

$$\Delta F = F_s - F_p \quad (30)$$

and serves as a measure of the width of the transition band of the filter.

It was shown earlier that the error curve for the optimal low-pass filter could have either $r + 1$ or $r + 2$ extrema where $r = (N + 1)/2$ for case 1, and $r = N/2$ for case 2. It is important to understand the nature of the optimum lowpass filter to see under what conditions the number of ripple extrema attains the maximum value. It has been found experimentally that a reasonably straightforward and informative way of summariz-

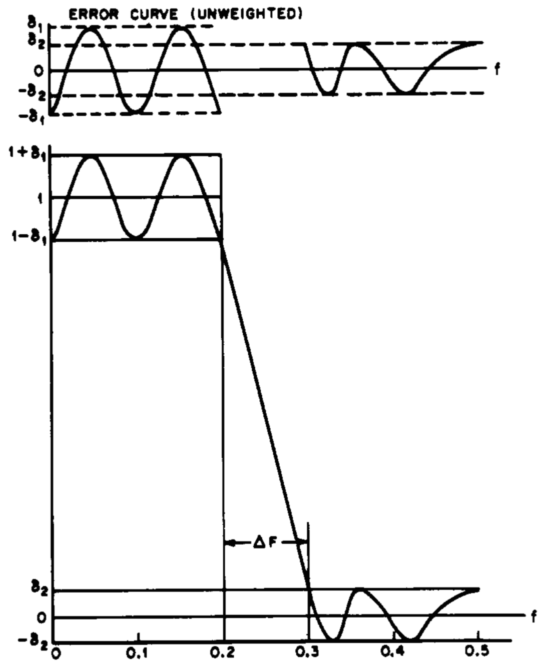


Fig. 7. Frequency response and error curve of optimal low-pass filter.

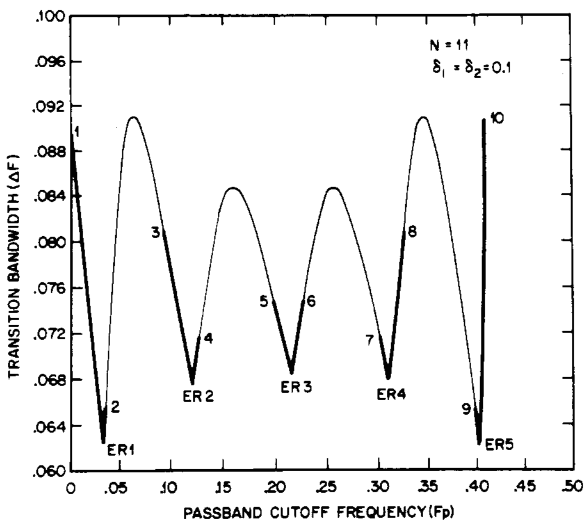


Fig. 8. The curve of transition width versus passband cutoff frequency for optimal low-pass filters.

ing the behavior of the optimum filter is to plot the transition width of the filter ($\Delta F$) versus passband cutoff frequency $F_p$, for fixed values of $N$, $\delta_1$, and $\delta_2$. Fig. 8 shows such a plot for case 1 data for $N = 11$, $\delta_1 = \delta_2 = 0.1$. As seen in this figure the curve of $\Delta F$ versus $F_p$ has an oscillatory behavior, alternating between sharp minima, and flat-topped maxima. The local minima of the curves (labelled *ER*1 to *ER*5) have been found to be the maximal ripple (extraripple) filters for the particular choice of $N$, $\delta_1$, and $\delta_2$. (Recall that extraripple filters have $(N + 5)/2$ equal amplitude extrema in their error curves.) There are exactly $(N - 1)/2$ of these extraripple filters. In between the extraripple solutions, it has been found that there are two types of optimum filters—scaled extraripple filters and equiripple filters with exactly $(N + 3)/2$ equal amplitude extrema in their error curves.

The scaled extraripple filters (shown in Fig. 8 by the heavy lines on the curve) have the property that their error curves

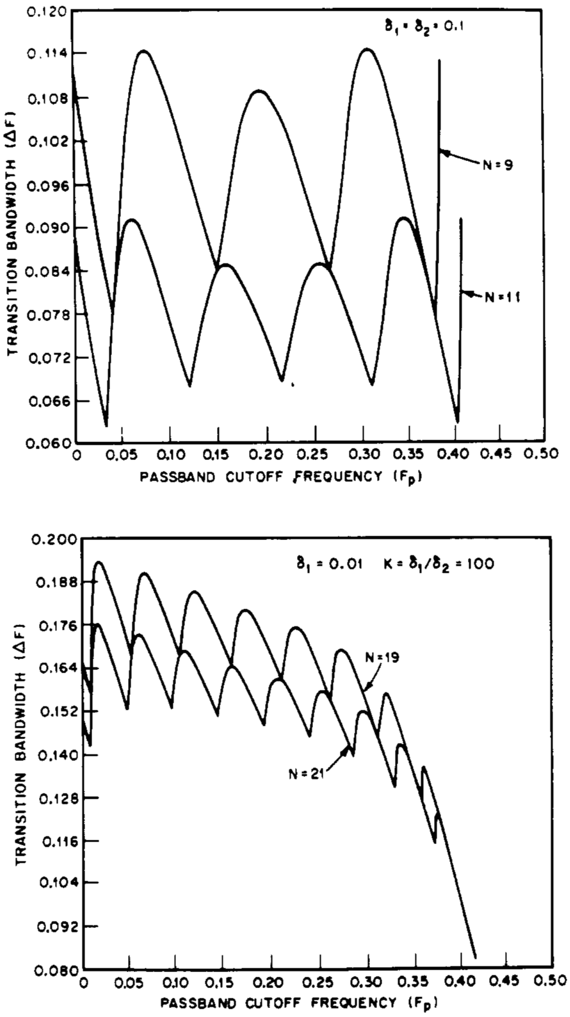


Fig. 9. The curves of transition width versus passband cutoff frequency for several values of low-pass filter parameters.

have $(N + 3)/2$ equal amplitude extrema, as well as one smaller amplitude extremum which is either at $f = 0$ or $f = 0.5$. These filters can be derived from their neighboring extraripple filter by a simple scaling procedure [25], [26].

The optimum filters between scaled extraripple solutions all have exactly $(N + 3)/2$ equal amplitude extrema in their error curves. No simple linear scaling procedure has been found to account for their presence.

However, an extraripple filter of length $N - 2$ has $(N + 3)/2$ equal amplitude extrema in its error curves and thus it is an optimal length $N$ filter (with the highest order coefficient zero). This point is evident in Fig. 9 which shows the curves of transition width versus passband cutoff frequency for $N = 9$ and 11, with $\delta_1 = \delta_2 = 0.1$ as well as for $N = 19$ and 21 with $\delta_1 = 0.001$ and $\delta_2 = 0.0001$. The curves touch at the local minima of the $N = 9$ curve at the top and at the local minima of the $N = 19$ curve at the bottom.

Fig. 10 presents a summary of the types of optimal filters which may be obtained by varying the filter cutoff frequencies. The first filter shown is an extraripple solution with $N = 25$, $\delta_1 = \delta_2 = 0.05$. Below it are two different scaled solutions where the frequency response is 0 at $f = 0.5$ for the first filter and then 0.03 at $f = 0.5$ for the second case. The last filter in the first column represents the maximum possible scaling, i.e., the frequency response is 0.05 at $f = 0.5$ with $(N + 3)/2$ equal

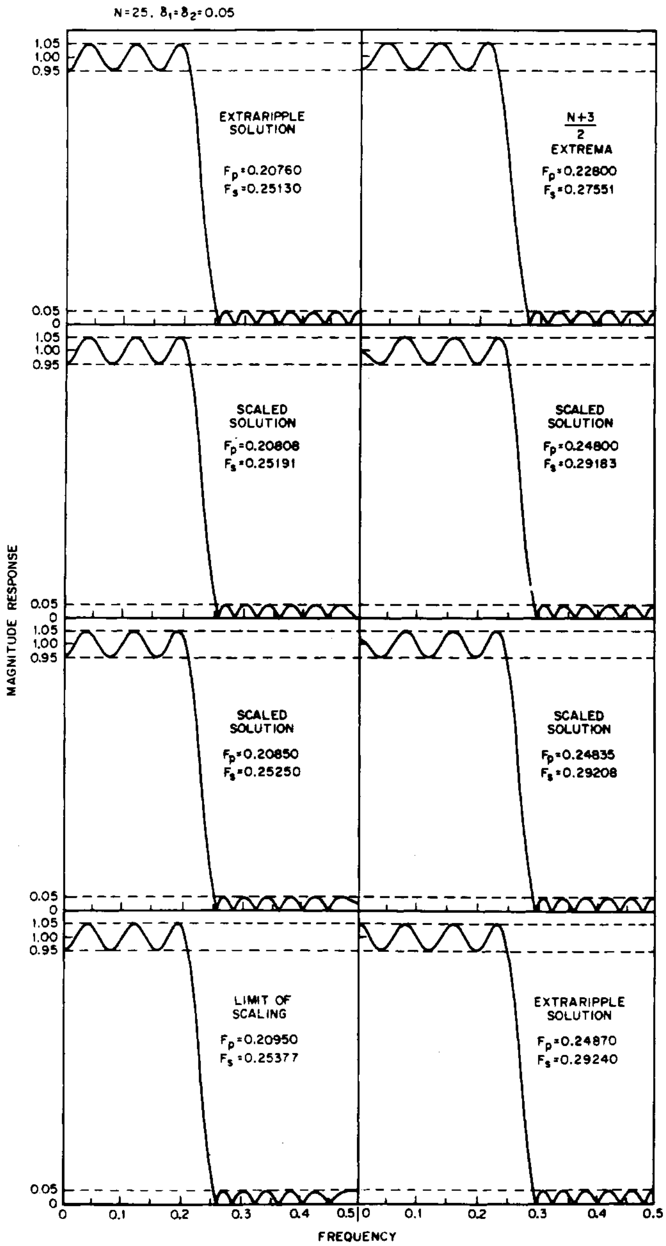


Fig. 10. Summary of the types of optimal low-pass filters.

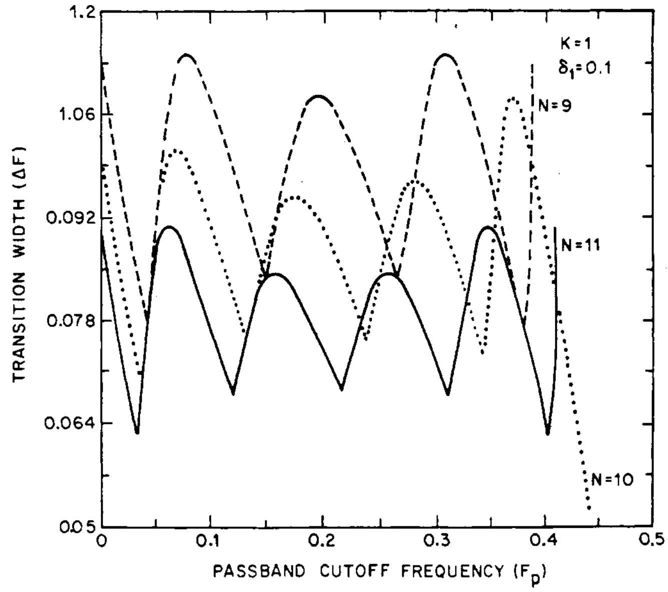


Fig. 11. A comparison of the curves of transition width versus passband cutoff frequency for low-pass filters with even and odd values of $N$.

amplitude extrema in the error curve. In the second column, the first filter is a point approximately midway between extraripple solutions. The next two filters represent scaled extraripple solutions where the error at $f = 0$ is not of the same value as the other error extrema. For the first of these filters the error is about $-0.005$ at $f = 0$, whereas for the second it is about 0.015 at $f = 0$. The last filter in the second column corresponds to the next extraripple solution.

## IX. Relations Between Optimal Low-Pass Filter Parameters

A great deal has been learned about the relationships between the parameters of optimal low-pass filters. In this section we summarize some of the key results.

### *A. Chebyshev Solutions [27]*

An analytical solution to the optimal filter design problem exists for the case of extraripple designs with either one passband, or one stopband ripple. Since these cases are either very wide-band, or very narrow-band designs, they are not generally of much interest, except for the insights they provide into analytical relations between the various filter parameters.

### *B. Symmetry Relations [27]*

A symmetry exists in the design parameters in the following sense. If $H(e^{j\omega})$ is the frequency response of an optimal low-pass filter with parameters $N$, $F_p$, $F_s$, $\delta_1$, and $\delta_2$, then $G(e^{j\omega}) = 1 - H(e^{j(\pi-\omega)})$ is the frequency response of another optimal low-pass filter with parameters $N$, $F_p' = 0.5 - F_s$, $F_s' = 0.5 - F_p$, $\delta_1' = \delta_2$, $\delta_2' = \delta_1$. This symmetry explains the behavior of the curve of $\Delta F$ versus $F_p$ of Fig. 8 since, in this case, $\delta_1 = \delta_2$; therefore, any filter with parameter $F_p$ has a symmetrical partner with parameter $0.5 - F_s$.

### *C. Case 2 Low-Pass Filters–N Even [29]*

An interesting design relation exists when comparing case 1 and case 2 low-pass filters. Fig. 11 shows a plot of $\Delta F$ versus $F_p$ for $\delta_1 = \delta_2 = 0.1$ and $N = 9$, 10, and 11. Notice that for certain values of $F_p$, the transition width is smaller for $N = 10$ designs than for $N = 9$; whereas for other values of $F_p$, the transition width is smaller for $N = 9$ designs than for $N = 10$. Thus monotonicity of transition width as a function of $N$ is not preserved across both even and odd values but instead holds only for comparing either $N$ odd designs, or $N$ even designs.

### *D. Design Formulas [27], [30]*

Although exact analytical relations do not exist between the 5 low-pass filter parameters, a set of approximate relations can be given which is valid to within some reasonable bounds. Kaiser has proposed the particularly simple formula

$$N \approx \frac{-20 \log_{10} (\sqrt{\delta_1 \delta_2}) - 13}{14.6\, \Delta F} + 1 \tag{31}$$

for predicting the filter length $N$ from the ripple specifications and the band edge frequencies. A somewhat more accurate formula due to Herrmann *et al.* [27] is

$$N \approx \frac{D_\infty(\delta_1, \delta_2) - f(\delta_1, \delta_2)(\Delta F)^2}{\Delta F} + 1 \tag{32}$$

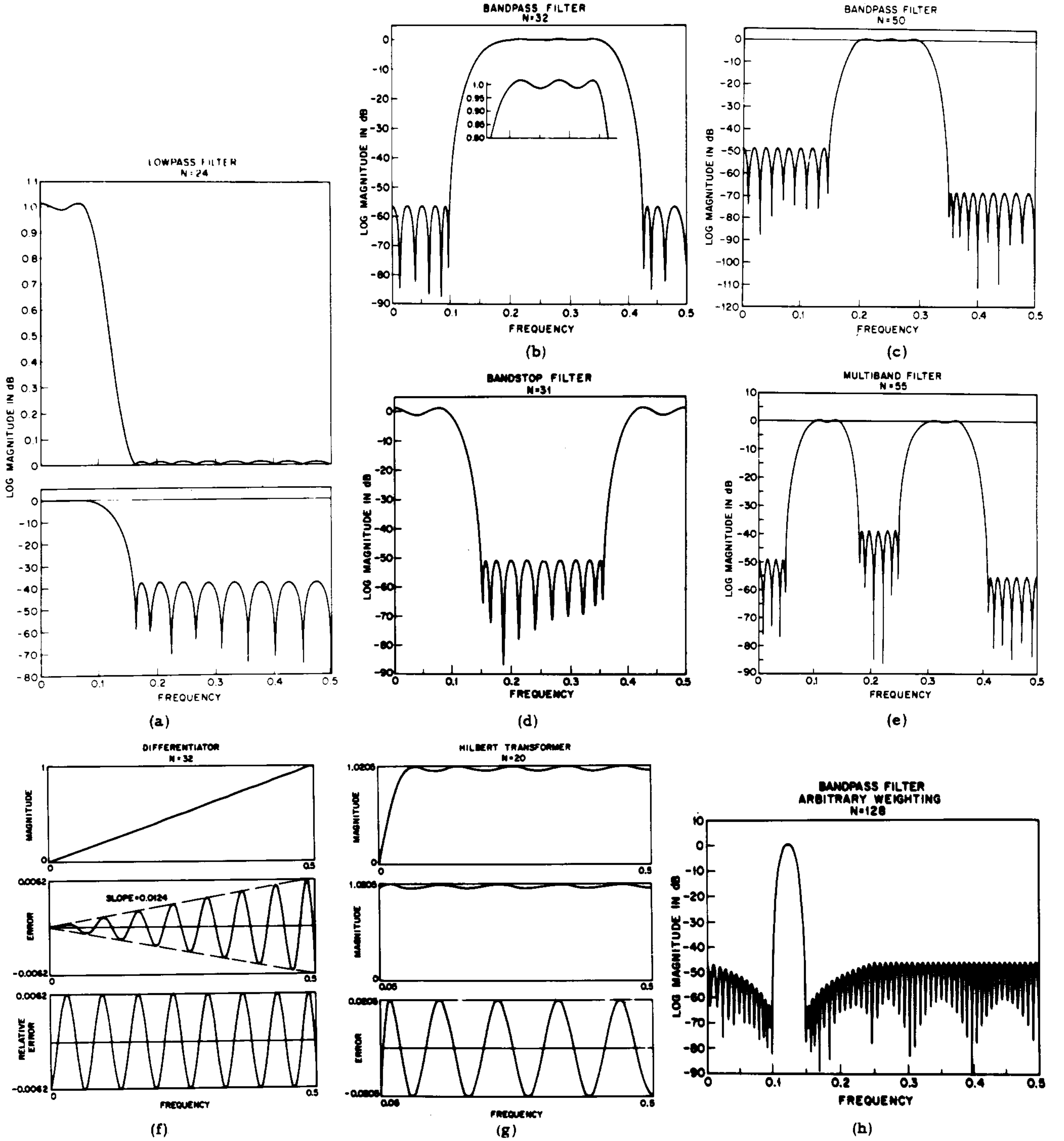


Fig. 12. A summary of several optimal FIR filters.

where

$$D_\infty = [0.005309 (\log_{10} \delta_1)^2 + 0.07114 (\log_{10} \delta_1) - 0.4761] \log_{10} \delta_2 - [0.00266 (\log_{10} \delta_1)^2 + 0.5941 \log_{10} \delta_1 + 0.4278] \tag{33}$$

and

$$f(\delta_1, \delta_2) = 11.012 + 0.51244 (\log_{10} \delta_1 - \log_{10} \delta_2). \tag{34}$$

Such design formulas have proved exceedingly useful for getting a good estimate of the required filter length.

## X. Examples of Optimal Filters

Fig. 12 illustrates some typical optimal filters which have been designed using the iterative approximation method of Section VI, and the program implementation of McClellan *et al.* These examples are meant to illustrate the power of the design method in approximating a wide range of filter types. Fig. 12(a) shows an $N = 25$ low-pass filter; Fig. 12(b) shows an $N = 32$ bandpass filter with equal sidelobe levels; Fig. 12(c) shows an $N = 50$ bandpass filter with unequal sidelobe levels; Fig. 12(d) shows an $N = 31$ bandstop filter; Fig. 12(e) shows an $N = 55$ multiband filter with 3 stopbands and 2 passbands; Fig. 12(f) shows an $N = 32$ equiripple relative error differentiator; Fig. 12(g) shows an $N = 20$ Hilbert transformer; and Fig. 12(h) shows an $N = 128$ bandpass filter with arbitrary weighting near the edges of the stopbands.

## XI. Design of Filters with Time and Frequency Domain Constraints

We have discussed the design of digital filters which approximate characteristics of a specified frequency response only. Quite often one would like to impose simultaneous restrictions on both the time and frequency response of the filter. For example, in the design of lowpass filters, one would often like to limit the step response overshoot or ripple; at the same time maintaining some reasonable control over the frequency response of the filter. Since the step response is a linear function of the impulse response coefficients, a linear program is capable of setting up constraints of the type discussed above. By way of example, we consider the design of a case 1 lowpass filter with the following specifications.

*Passband:*

$$1 - \delta_1 \leq H^*(e^{j\omega}) \leq 1 + \delta_1. \tag{35}$$

*Stopband:*

$$-\delta_2 \leq H^*(e^{j\omega}) \leq \delta_2. \tag{36}$$

*Step Response:*

$$-\delta_3 \leq g(n) \leq \delta_3 \tag{37}$$

where $g(n)$ is the step response of the filter, and is defined as

$$g(n) = \sum_{m=0}^{n} h(m). \tag{38}$$

Clearly $g(n)$ is a linear combination of the impulse response coefficients; hence (33)–(35) can be solved using linear programming techniques on the deltas. For example, one could fix any one or two of the parameters $\delta_1$, $\delta_2$, or $\delta_3$ and minimize the other(s). Alternatively one could set $\delta_1 = \alpha_1\delta$, $\delta_2 = \alpha_2\delta$, and $\delta_3 = \alpha_3\delta$ where $\alpha_1$, $\alpha_2$, and $\alpha_3$ are constants, and simultaneously minimize all three parameters by minimizing $\delta$.

Another application is in the design of interpolation filters [28] where some of the impulse response coefficients must be constrained to be zero. In this case the Alternation Theorem no longer applies because the basis functions of the approximation do not satisfy the Haar condition. Thus, the Remez exchange algorithm cannot be used to calculate the best approximation. But since linear programming does not depend on the Haar condition for its convergence, it can and has been applied to this problem.

## XII. Filters with Optimum Magnitude and Minimum Phase

There may be applications where the linear phase characteristic may not be necessary and one is only interested in the shortest possible filter length for a given magnitude response. In such cases a natural question is: how much can the filter length be reduced for a given $F_p$, $F_s$, $\delta_1$, and $\delta_2$ by dropping the requirement for linear phase? Since half of the filter coefficients are constrained by the symmetry required for linear phase as shown in Fig. 1, one might at first guess that by dropping the linear phase requirement the required filter length might be cut in half. This is generally not the case. In fact the length reduction to be expected depends very much on the

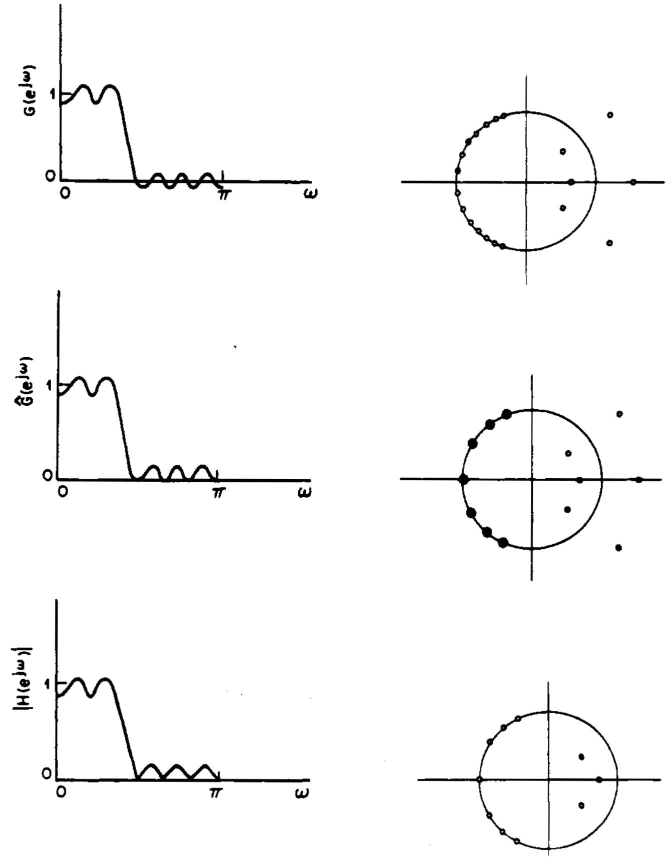


Fig. 13. The procedure for obtaining the optimal magnitude low-pass filter from an optimal linear phase design.

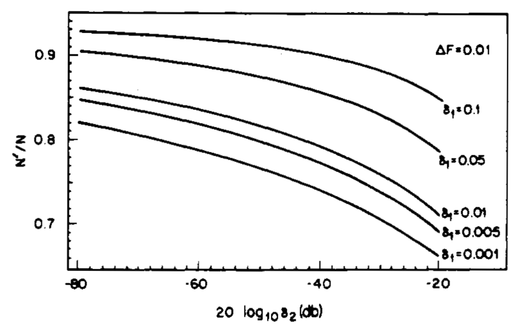


Fig. 14. Ratio of $N'$, the length of the optimum magnitude filter, to $N$, the length of the optimum linear phase filter, with identical low-pass parameters.

type of filter under consideration as will be described in this section.

First, let us define what we mean by an optimum magnitude approximation. For a given desired magnitude $D(e^{j\omega})$ and weight function $W(e^{j\omega})$, the optimum magnitude approximation is the filter which minimizes

$$\max_{\omega \in \Omega} W(e^{j\omega}) \left| D(e^{j\omega}) - |H(e^{j\omega})| \right|$$

where $\Omega$ is the union of the frequency bands of interest.

For the case where $D(e^{j\omega})$ is piecewise constant over the frequency bands of interest (e.g., a low-pass or bandpass filter), a procedure suggested by Herrmann and Schussler [17] will yield the optimum magnitude filter. Briefly, this procedure works as follows for the lowpass case. A weighted Chebyshev approximation problem is solved for a linear combination of $N$ cosines by using the Remez algorithm described

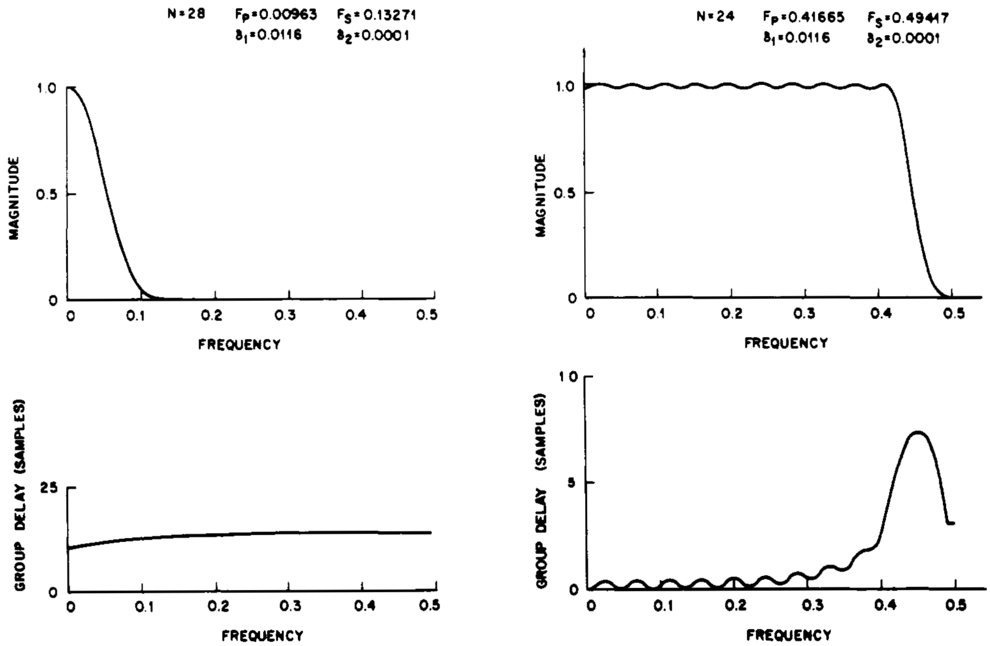


Fig. 15. The magnitude and group delay responses for a narrow-band and a wide-band optimum magnitude filter.

earlier to design a length $2N-1$ linear phase filter (see Fig. 13(a)). The response $G(e^{j\omega})$ is then scaled (by addition of the constant $\delta_2$ to the frequency response) to give a $\hat{G}(e^{j\omega})$ which is positive with double zeros in the stopband as shown in Fig. 13(b). The resulting order $2N-2$ mirror image polynomial is factored retaining one each of the double zeros on the unit circle and all zeros inside the unit circle. Finally, the filter is scaled so that in the passband the filter response approximates one as shown in Fig. 13(c). The resulting length $N$ filter is the optimum magnitude approximation and is also a minimum phase transfer function [17].

To see how much the filter length can be reduced from the linear phase design, while retaining the same magnitude characteristics, the design formula (30) can be used to calculate $N'/N$, the ratio of lengths for optimum squared magnitude and linear phase filters.

Fig. 14 shows how this calculated improvement depends on the other filter parameters. Since (30) does not apply to either very narrow or very wide-band filters, a separate analysis is required. In fact it can be shown that for very narrow-band filters where all zeros are on the unit circle of the $z$ plane the optimum squared magnitude filter is a linear phase filter and $N'/N = 1$. For the other extreme of very wide-band filters one should expect $N'/N = 1/2$.

The actual realization of these optimum magnitude filters will be less efficient (in terms of the number of multiplications per sample in a direct form realization) than a linear phase filter with similar performance. This is due to the fact that the symmetry of the linear phase impulse response will allow a 2 to 1 reduction in the number of multiplies and this will offset the 10- or 20-percent reduction in length of the optimum magnitude filters. In cases where one is interested in minimizing the required number of delays, optimum magnitude filters become more attractive.

The phase characteristics of the optimum magnitude filters also depend very much on the filter bandwidth. As the bandwidth is increased from the very narrow-band optimum magnitude filter with linear phase (constant group delay), the group delay characteristic deteriorates until for very wide band filters the group delay is reminiscent of that for an elliptic filter with a sharp peak near the band edge. To illustrate this point, Fig. 15 shows examples of the magnitude and group delay response for a narrow-band and a wide-band low-pass filter.

While this section has focused on low-pass filters for simplicity, the procedure for designing optimum magnitude filters can also be applied to the general bandpass filter case with several pass and stop bands. The improvement to be expected by dropping the linear phase requirement is very small for filters with narrow passbands but much larger for filters with narrow stopbands.

A further generalization of the design of linear phase filters would allow the simultaneous approximation of both magnitude and phase. Such filters could find applications as phase equalizers. However, at this time extensions of the Chebyshev theory to this case have met with little success [35]. One promising algorithm which has been used for the design of filters with both magnitude and phase specifications is the Lawson algorithm [44].

## XIII. The Two-Dimensional FIR Approximation Problem

The approximation problem for two-dimensional FIR digital filters is a much more difficult problem than the corresponding one-dimensional design problem. Some of the one-dimensional filter design techniques have been extended to two dimensions [37], [38], but for other techniques such a generalization appears unlikely. In particular, the iterative design methods based on the Remez exchange algorithm have not been extended to the two-dimensional case, and so at this time there is no efficient procedure available in two dimensions for designing optimal Chebyshev filters.[3]

There are two reasons why the optimal algorithm based on the alternation theorem cannot be extended to two dimen-

[3] Recently, work on exchange algorithms in two dimensions has been carried out by Kamp and Thiran [41], and Hersey and Mersereau [42].

sions. First, it is impossible for any set of functions defined on a two-dimensional domain to satisfy the Haar condition. Thus, the alternation theorem applies in a weaker form. Secondly, there is no possibility of ordering the extremal frequencies as in the one-dimensional case, where increasing ordering guarantees that the error changes sign from one point to the next. Even if the method could be extended, the size of the problem is a handicap. For example, the design of an optimum 31 X 31 FIR linear phase filter involves optimization over 16 X 16 = 256 parameters.

Since linear programming does not depend on the Haar condition, it can be applied to the two-dimensional approximation problem. While convergence of the linear program is guaranteed, the size of the problem (proportional to $N^2$) and the inefficiency of the linear programming technique has limited this technique to the design of low-order filters, e.g., 9 X 9 is the largest reported in Hu and Rabiner [38].

The suboptimal techniques of windowing and frequency sampling have also been extended to two dimensions and are capable of designing higher order filters.

In view of the computational difficulties of linear programming and the lack of theory for an exchange algorithm, two-dimensional approximation techniques have concentrated on suboptimal techniques. Included in the class of suboptimal methods are the windowing technique, the frequency sampling technique and an approach based on transformations of one-dimensional filters. This transformation technique in some cases may yield an optimal filter. In this section we outline the basic idea for the transformation method of design.

A two-dimensional digital filter with impulse response matrix $\{h(k,p)\}, k = 0, 1, \cdots, N_1 - 1; p = 0, 1, \cdots, N_2 - 1$ has a frequency response defined by the two-dimensional Fourier transform

$$H(e^{j\omega_1}, e^{j\omega_2}) = \sum_{k=0}^{N_1-1} \sum_{p=0}^{N_2-1} h(k,p) \exp\left[-j(k\omega_1 + p\omega_2)\right]. \tag{39}$$

If the impulse response is constrained to be symmetric,

$$\begin{aligned} h(N_1 - m - 1, p) &= h(m,p), \qquad m = 0, 1, \cdots, n_1 = (N_1 - 1)/2 \\ h(k, N_2 - m - 1) &= h(k,m), \qquad m = 0, 1, \cdots, n_2 = (N_2 - 1)/2 \end{aligned} \tag{40}$$

then the frequency response can be rewritten as

$$H(e^{j\omega_1}, e^{j\omega_2}) = \exp\left[-j(n_1\omega_1 + n_2\omega_2)\right] \hat{H}(\omega_1, \omega_2) \tag{41}$$

where

$$\hat{H}(e^{j\omega_1}, e^{j\omega_2}) = \sum_{k=0}^{n_1} \sum_{p=0}^{n_2} a(k,p) \cos k\omega_1 \cos p\omega_2. \tag{42}$$

Since $\exp[-j(n_1\omega_1 + n_2\omega_2)]$ has magnitude one, $|\hat{H}(e^{j\omega_1}, e^{j\omega_2})|$ is the magnitude of the frequency response. The folding frequency along each axis is $\pi$, so the magnitude response is completely specified on the square $[0, \pi] \times [0, \pi]$.

Recall the type-1 one-dimensional filter where $G(e^{j\omega})$ can be written in the form

$$G(e^{j\omega}) = \sum_{n=0}^{M} b(n) \cos(n\omega) \tag{43}$$

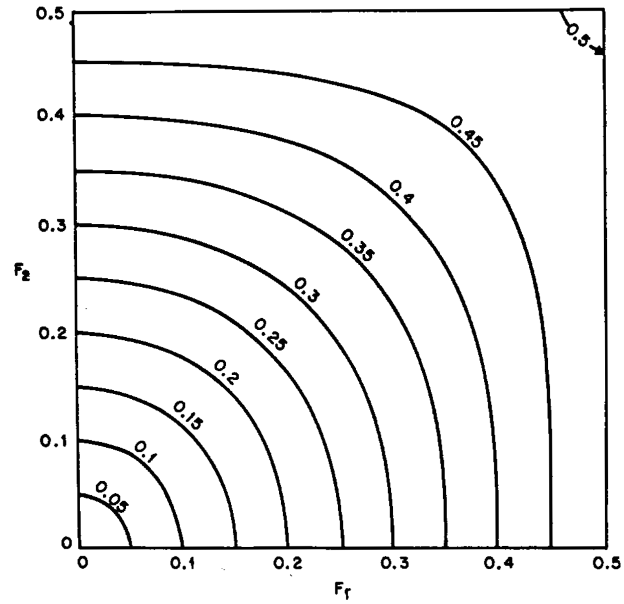


Fig. 16. Contours of the mapping in frequency from one to two dimensions.

where $M = (N-1)/2$, $b(0) = h(M)$, and $b(n) = 2h(M-n)$, $n = 1, 2, \cdots, M$. The magnitude of the frequency response is $|G(e^{j\omega})|$.

If the change of variables

$$\cos\omega = A\cos\omega_1 + B\cos\omega_2 + C\cos\omega_1\cos\omega_2 + D \tag{44}$$

is made in $G(e^{j\omega})$ then a function of the form $\hat{H}(e^{j\omega_1}, e^{j\omega_2})$ results. Thus the one-dimensional response $G(e^{j\omega})$ is mapped to a two-dimensional response $\hat{H}(e^{j\omega_1}, e^{j\omega_2})$. Given a fixed value of $\omega \in [0, \pi]$ there corresponds a curve in the $(\omega_1, \omega_2)$ plane, and along this curve the transformed two-dimensional frequency response is a constant equal to the value of the one-dimensional frequency response at $\omega$. As $\omega$ varies, a family of contours is generated which completely describes the transformed two-dimensional frequency response. For the choice of parameters $A = B = C = -D = \frac{1}{2}$, the contours are shown in Fig. 16. Thus a low-pass filter will be mapped by this particular change of variables to a low-pass circularly symmetric two-dimensional filter.

An important feature of this new method is that it is not limited to the design of small filters. With a larger filter it is possible to obtain smaller deviations and/or a sharper cutoff. Fig. 17 shows the magnitude of the frequency response for a 31 X 31 circularly symmetric lowpass filter with a transition region of width $0.10\pi$. The design time for this filter was approximately 5 s on an IBM 370/155 computer.

In addition, it can be shown that the filter of Fig. 17 is an optimal filter, if one is willing to accept the approximately circular contours shown in Fig. 16.

The possibilities for new design algorithms for two-dimensional filters are still wide open but for the present, suboptimal techniques are the only ways to get large filters.

## XIV. Summary

The mathematical theory of Chebyshev approximation has been used as the unifying theme in the presentation of recent design algorithms for FIR digital filters. Special emphasis was placed on the linear phase design problem where the methods are highly developed due to the direct applicability of the

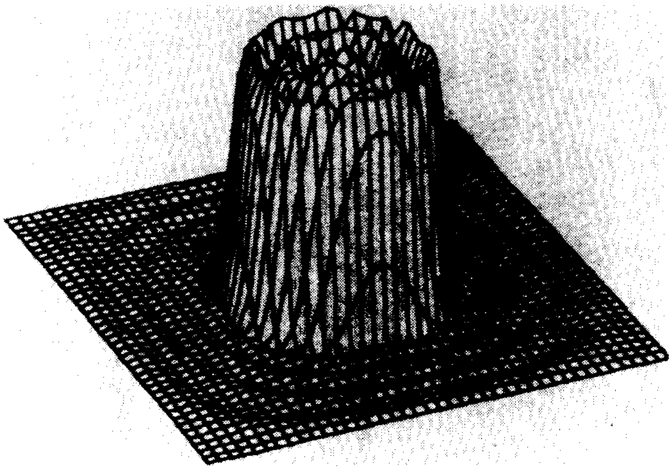

Fig. 17. Frequency response of a 31 X 31 low-pass filter.

Chebyshev theory. In the more general cases of magnitude approximation and two-dimensional approximation, it was shown that special techniques can be used to obtain optimal low-pass or bandpass filters.

Within the realm of Chebyshev approximation of FIR digital filters several research problems remain. These include the simultaneous approximation of magnitude and phase, general two-dimensional approximation, and the optimization of the filter response under the constraint of finite wordlength coefficients.

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# Two-Dimensional Digital Filtering

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*Invited Paper*

***Abstract*—The problems of designing and implementing LSI systems for the processing of 2-*D* digital data, such as images or geophone arrays, are reviewed and discussed. This discussion encompasses both FIR and IIR digital filters and with respect to the latter, the issues of stability testing and filter stabilization are also considered. Techniques are also presented whereby such filtering can be accomplished using either 1 or 2-*D* LSI systems.**

## I. Introduction

OUR OBJECTIVE in this paper is to review the mathematical framework underlying the two-dimensional (2-*D*) digital filtering problem and to explore available techniques for the design and implementation of 2-*D* linear shift-invariant (LSI) digital filters. In the course of this discussion, the many similarities to and differences from the one-dimensional (1-*D*) digital filtering problem will hopefully become apparent. Restricting our attention to studying LSI systems allows us to use the powerful techniques of Fourier analysis which have proved their value in numerous problems of practical interest in both 1 and 2 dimensions.

Other papers in this special issue [1]-[3] have touched upon the need for 2-*D* digital filters within the larger context of 2-*D* digital signal processing. Such filters are central to many image and array processing applications—such as X-ray enhancement, image deblurring, scene analysis, weather predictions, seismic analysis, and the processing of radar and sonar arrays, to name just a few. The discussions in this paper, while applicable to any of these problems, will not be specifically directed toward any one of them.

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1-*D* LSI systems represent a special case of two and higher dimensional systems. Therefore, many 2-*D* concepts will look vaguely familiar. For this reason, those 2-*D* results which are straightforward extensions of 1-*D* results will be presented without extensive discussions. On the other hand, there are many properties of 1-*D* LSI systems which cannot be easily generalized, which is why 2-*D* digital signal processing remains a challenging and interesting field of study. These difficulties are almost always related to the fact that there is no fundamental theorem of algebra for polynomials in two independent variables [4]. The reader will recall that it is this theorem which allows us to factor a 1-*D* polynomial of degree *n* into a product of *n* polynomial factors of first degree, thereby allowing us to find the roots of polynomials, to check the stability of a filter by finding the locations of the poles of its system function, and to realize digital filters as cascade or parallel structures [5].

Because there are many operations which are more easily performed using 1-*D* mathematics, it is perhaps worthwhile to use 1-*D* filters to perform 2-*D* tasks. Interestingly this can be done, and some techniques will be presented later in this paper. While the ultimate value of such an approach remains to be established, it is the authors' belief that this approach will prove to be quite useful both conceptually as well as in practice. Inasmuch as this material is fairly new, it has received a rather heavy emphasis in this paper.

The remainder of this paper is divided into four parts. In Section II, a number of definitions are presented, primarily to establish notation and to serve as a reference for later sections. The representation of 2-*D* arrays by 1-*D* sequences is also discussed here. In Section III, we discuss the related issues of stability testing and the stabilization of unstable

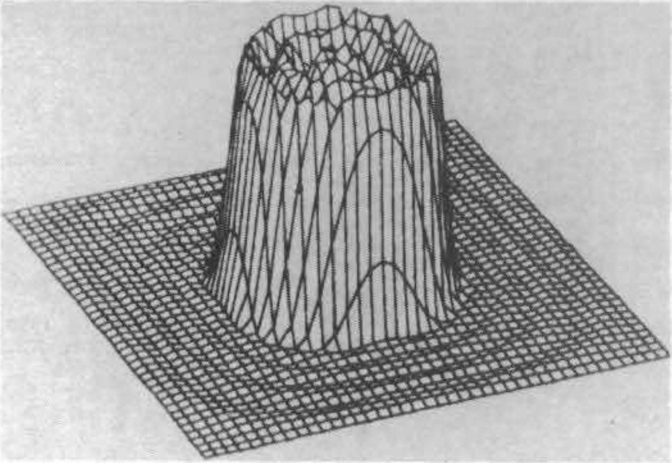

Fig. 17. Frequency response of a 31 X 31 low-pass filter.